6 | APPLICATIONS OF INTEGRATION

Figure 6.1 Hoover Dam is one of the United States' iconic landmarks, and provides irrigation and hydroelectric power for millions of people in the southwest United States. (credit: modification of work by Lynn Betts, Wikimedia)

Chapter Outline

Introduction

The Hoover Dam is an engineering marvel. When Lake Mead, the reservoir behind the dam, is full, the dam withstands a great deal of force. However, water levels in the lake vary considerably as a result of droughts and varying water demands. Later in this chapter, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force (see **Example 6.28**).

Hydrostatic force is only one of the many applications of definite integrals we explore in this chapter. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us.

6.1 | Areas between Curves

Learning Objectives

6.1.1 Determine the area of a region between two curves by integrating with respect to the independent variable.

6.1.2 Find the area of a compound region.

6.1.3 Determine the area of a region between two curves by integrating with respect to the dependent variable.

In **Introduction to Integration**, we developed the concept of the definite integral to calculate the area below a curve on a given interval. In this section, we expand that idea to calculate the area of more complex regions. We start by finding the area between two curves that are functions of x, beginning with the simple case in which one function value is always greater than the other. We then look at cases when the graphs of the functions cross. Last, we consider how to calculate the area between two curves that are functions of y.

Area of a Region between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ such that $f(x) \geq g(x)$ on $[a, b]$. We want to find the area between the graphs of the functions, as shown in the following figure.

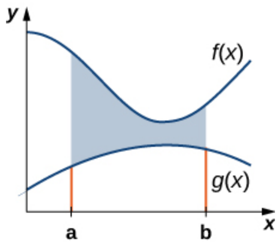


Figure 6.2 The area between the graphs of two functions, $f(x)$ and $g(x)$, on the interval $[a, b]$.

As we did before, we are going to partition the interval on the x-axis and approximate the area between the graphs of the functions with rectangles. So, for $i = 0, 1, 2, \ldots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$, and on each interval $[x_{i-1}, x_i]$ construct a rectangle that extends vertically from $g(x_i^*)$ to $f(x_i^*)$. **Figure 6.3**(a) shows the rectangles when x_i^* is selected to be the left endpoint of the interval and $n = 10$. **Figure 6.3**(b) shows a representative rectangle in detail.

Use this **calculator (http://www.openstax.org/l/20_CurveCalc)** to learn more about the areas between two curves.

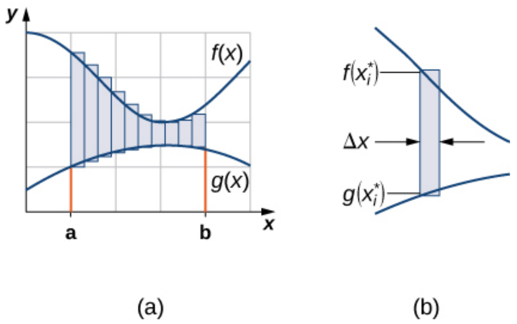


Figure 6.3 (a)We can approximate the area between the graphs of two functions, $f(x)$ and $g(x)$, with rectangles. (b) The area of a typical rectangle goes from one curve to the other.

The height of each individual rectangle is $f(x_i^*) - g(x_i^*)$ and the width of each rectangle is Δx. Adding the areas of all the rectangles, we see that the area between the curves is approximated by

$$A \approx \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x.$$

This is a Riemann sum, so we take the limit as $n \to \infty$ and we get

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x = \int_a^b [f(x) - g(x)]dx.$$

These findings are summarized in the following theorem.

Theorem 6.1: Finding the Area between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

$$A = \int_a^b [f(x) - g(x)]dx. \tag{6.1}$$

We apply this theorem in the following example.

Example 6.1

Finding the Area of a Region between Two Curves 1

If R is the region bounded above by the graph of the function $f(x) = x + 4$ and below by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval $[1, 4]$, find the area of region R.

Solution

The region is depicted in the following figure.

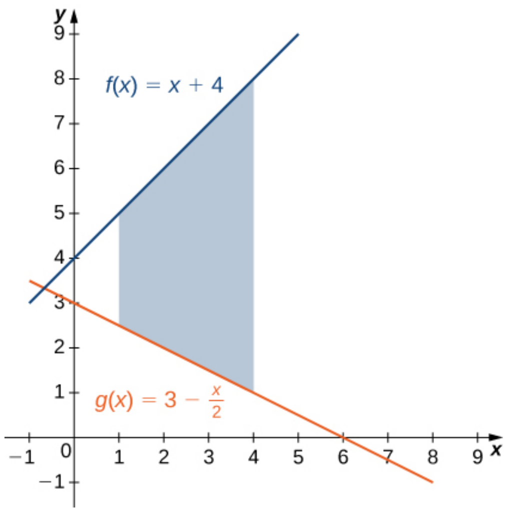


Figure 6.4 A region between two curves is shown where one curve is always greater than the other.

We have

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]dx \\ &= \int_1^4 \left[(x+4) - \left(3 - \frac{x}{2}\right)\right]dx = \int_1^4 \left[\frac{3x}{2} + 1\right]dx \\ &= \left[\frac{3x^2}{4} + x\right]\Big|_1^4 = \left(16 - \frac{7}{4}\right) = \frac{57}{4}. \end{aligned}$$

The area of the region is $\frac{57}{4}$ units^2.

6.1 If R is the region bounded by the graphs of the functions $f(x) = \frac{x}{2} + 5$ and $g(x) = x + \frac{1}{2}$ over the interval $[1, 5]$, find the area of region R.

In **Example 6.1**, we defined the interval of interest as part of the problem statement. Quite often, though, we want to define our interval of interest based on where the graphs of the two functions intersect. This is illustrated in the following example.

Example 6.2

Finding the Area of a Region between Two Curves 2

If R is the region bounded above by the graph of the function $f(x) = 9 - (x/2)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region R.

Solution

The region is depicted in the following figure.

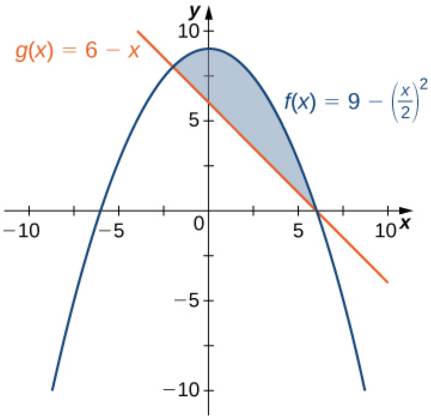


Figure 6.5 This graph shows the region below the graph of $f(x)$ and above the graph of $g(x)$.

We first need to compute where the graphs of the functions intersect. Setting $f(x) = g(x)$, we get

$$\begin{aligned} f(x) &= g(x) \\ 9 - \left(\frac{x}{2}\right)^2 &= 6 - x \\ 9 - \frac{x^2}{4} &= 6 - x \\ 36 - x^2 &= 24 - 4x \\ x^2 - 4x - 12 &= 0 \\ (x-6)(x+2) &= 0. \end{aligned}$$

The graphs of the functions intersect when $x = 6$ or $x = -2$, so we want to integrate from -2 to 6. Since $f(x) \geq g(x)$ for $-2 \leq x \leq 6$, we obtain

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]dx \\ &= \int_{-2}^{6} \left[9 - \left(\frac{x}{2}\right)^2 - (6 - x)\right]dx = \int_{-2}^{6} \left[3 - \frac{x^2}{4} + x\right]dx \\ &= \left[3x - \frac{x^3}{12} + \frac{x^2}{2}\right]\Bigg|_{-2}^{6} = \frac{64}{3}. \end{aligned}$$

The area of the region is $64/3$ units^2.

6.2 If R is the region bounded above by the graph of the function $f(x) = x$ and below by the graph of the function $g(x) = x^4$, find the area of region R.

Areas of Compound Regions

So far, we have required $f(x) \geq g(x)$ over the entire interval of interest, but what if we want to look at regions bounded by the graphs of functions that cross one another? In that case, we modify the process we just developed by using the absolute value function.

Theorem 6.2: Finding the Area of a Region between Curves That Cross

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$. Let R denote the region between the graphs of $f(x)$ and $g(x)$, and be bounded on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

$$A = \int_a^b |f(x) - g(x)|dx.$$

In practice, applying this theorem requires us to break up the interval $[a, b]$ and evaluate several integrals, depending on which of the function values is greater over a given part of the interval. We study this process in the following example.

Example 6.3

Finding the Area of a Region Bounded by Functions That Cross

If R is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, \pi]$, find the area of region R.

Solution

The region is depicted in the following figure.

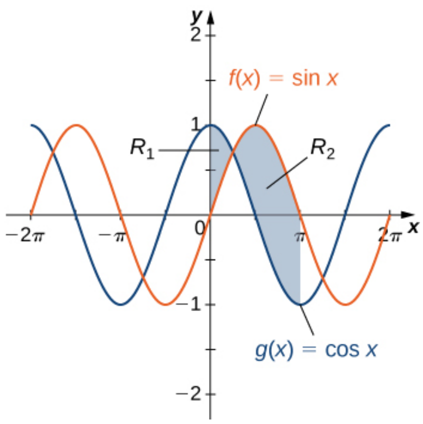


Figure 6.6 The region between two curves can be broken into two sub-regions.

The graphs of the functions intersect at $x = \pi/4$. For $x \in [0, \pi/4]$, $\cos x \geq \sin x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \cos x - \sin x.$$

On the other hand, for $x \in [\pi/4, \pi]$, $\sin x \geq \cos x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \sin x - \cos x.$$

Then

$$\begin{aligned} A &= \int_a^b |f(x) - g(x)| dx \\ &= \int_0^{\pi} |\sin x - \cos x| dx = \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx \\ &= [\sin x + \cos x] \big|_0^{\pi/4} + [-\cos x - \sin x] \big|_{\pi/4}^{\pi} \\ &= (\sqrt{2} - 1) + \left(1 + \sqrt{2}\right) = 2\sqrt{2}. \end{aligned}$$

The area of the region is $2\sqrt{2}$ units^2.

6.3 If R is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[\pi/2, 2\pi]$, find the area of region R.

Example 6.4

Finding the Area of a Complex Region

Consider the region depicted in **Figure 6.7**. Find the area of R.

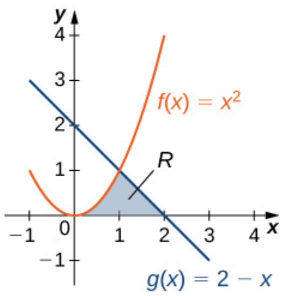


Figure 6.7 Two integrals are required to calculate the area of this region.

Solution

As with **Example 6.3**, we need to divide the interval into two pieces. The graphs of the functions intersect at $x = 1$ (set $f(x) = g(x)$ and solve for x), so we evaluate two separate integrals: one over the interval $[0, 1]$ and one over the interval $[1, 2]$.

Over the interval $[0, 1]$, the region is bounded above by $f(x) = x^2$ and below by the x-axis, so we have

$$A_1 = \int_0^1 x^2 dx = \frac{x^3}{3}\bigg|_0^1 = \frac{1}{3}.$$

Over the interval $[1, 2]$, the region is bounded above by $g(x) = 2 - x$ and below by the x-axis, so we have

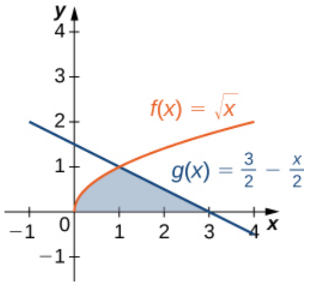
$$A_2 = \int_1^2 (2 - x)dx = \left[2x - \frac{x^2}{2}\right]\Big|_1^2 = \frac{1}{2}.$$

Adding these areas together, we obtain

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

The area of the region is $5/6$ units^2.

6.4 Consider the region depicted in the following figure. Find the area of R.



Regions Defined with Respect to *y*

In **Example 6.4**, we had to evaluate two separate integrals to calculate the area of the region. However, there is another approach that requires only one integral. What if we treat the curves as functions of y, instead of as functions of x? Review **Figure 6.7**. Note that the left graph, shown in red, is represented by the function $y = f(x) = x^2$. We could just as easily solve this for x and represent the curve by the function $x = v(y) = \sqrt{y}$. (Note that $x = -\sqrt{y}$ is also a valid representation of the function $y = f(x) = x^2$ as a function of y. However, based on the graph, it is clear we are interested in the positive square root.) Similarly, the right graph is represented by the function $y = g(x) = 2 - x$, but could just as easily be represented by the function $x = u(y) = 2 - y$. When the graphs are represented as functions of y, we see the region is bounded on the left by the graph of one function and on the right by the graph of the other function. Therefore, if we integrate with respect to y, we need to evaluate one integral only. Let's develop a formula for this type of integration.

Let $u(y)$ and $v(y)$ be continuous functions over an interval $[c, d]$ such that $u(y) \geq v(y)$ for all $y \in [c, d]$. We want to find the area between the graphs of the functions, as shown in the following figure.

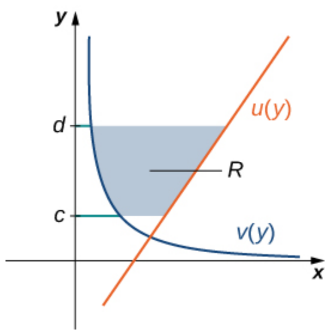


Figure 6.8 We can find the area between the graphs of two functions, $u(y)$ and $v(y)$.

This time, we are going to partition the interval on the y-axis and use horizontal rectangles to approximate the area between the functions. So, for $i = 0, 1, 2, \ldots, n,$ let $Q = \{y_i\}$ be a regular partition of $[c, d]$. Then, for $i = 1, 2, \ldots, n,$ choose a point $y_i^* \in [y_{i-1}, y_i],$ then over each interval $[y_{i-1}, y_i]$ construct a rectangle that extends horizontally from $v(y_i^*)$ to $u(y_i^*)$. **Figure 6.9**(a) shows the rectangles when y_i^* is selected to be the lower endpoint of the interval and $n = 10.$ **Figure 6.9**(b) shows a representative rectangle in detail.

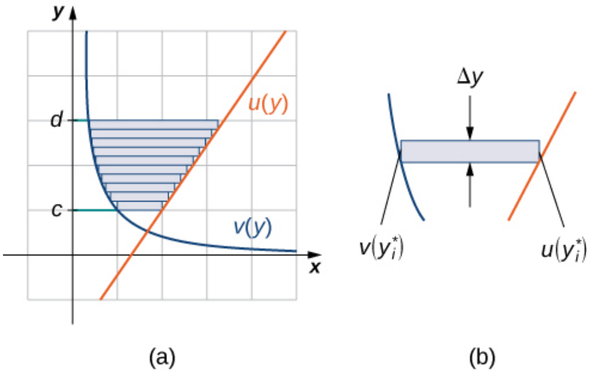


Figure 6.9 (a) Approximating the area between the graphs of two functions, $u(y)$ and $v(y),$ with rectangles. (b) The area of a typical rectangle.

The height of each individual rectangle is Δy and the width of each rectangle is $u(y_i^*) - v(y_i^*)$. Therefore, the area between the curves is approximately

$$A \approx \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)]\Delta y.$$

This is a Riemann sum, so we take the limit as $n \to \infty,$ obtaining

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)]\Delta y = \int_c^d [u(y) - v(y)]dy.$$

These findings are summarized in the following theorem.

Theorem 6.3: Finding the Area between Two Curves, Integrating along the *y*-axis

Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geq v(y)$ for all $y \in [c, d]$. Let R denote the region bounded on the right by the graph of $u(y),$ on the left by the graph of $v(y),$ and above and below by the lines $y = d$ and $y = c,$ respectively. Then, the area of R is given by

$$A = \int_c^d [u(y) - v(y)]dy. \tag{6.2}$$

Example 6.5

Integrating with Respect to *y*

Let's revisit **Example 6.4**, only this time let's integrate with respect to y. Let R be the region depicted in **Figure 6.10**. Find the area of R by integrating with respect to y.

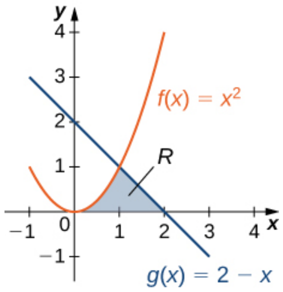


Figure 6.10 The area of region R can be calculated using one integral only when the curves are treated as functions of y.

Solution

We must first express the graphs as functions of y. As we saw at the beginning of this section, the curve on the left can be represented by the function $x = v(y) = \sqrt{y}$, and the curve on the right can be represented by the function $x = u(y) = 2 - y$.

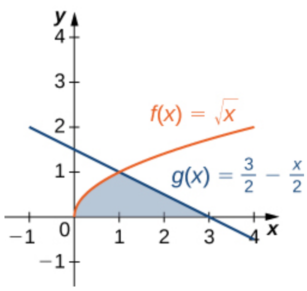
Now we have to determine the limits of integration. The region is bounded below by the x-axis, so the lower limit of integration is $y = 0$. The upper limit of integration is determined by the point where the two graphs intersect, which is the point $(1, 1)$, so the upper limit of integration is $y = 1$. Thus, we have $[c, d] = [0, 1]$.

Calculating the area of the region, we get

$$\begin{aligned} A &= \int_c^d [u(y) - v(y)]dy \\ &= \int_0^1 [(2 - y) - \sqrt{y}]dy = \left[2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2}\right]\Bigg|_0^1 \\ &= \frac{5}{6}. \end{aligned}$$

The area of the region is $5/6$ units^2.

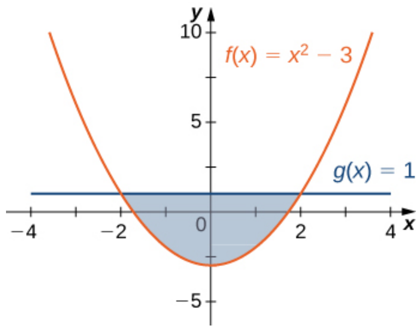
6.5 Let's revisit the checkpoint associated with **Example 6.4**, only this time, let's integrate with respect to y. Let R be the region depicted in the following figure. Find the area of R by integrating with respect to y.



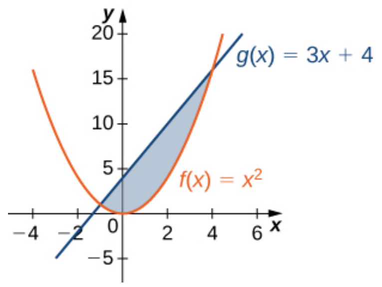
6.1 EXERCISES

For the following exercises, determine the area of the region between the two curves in the given figure by integrating over the x-axis.

1. $y = x^2 - 3$ and $y = 1$

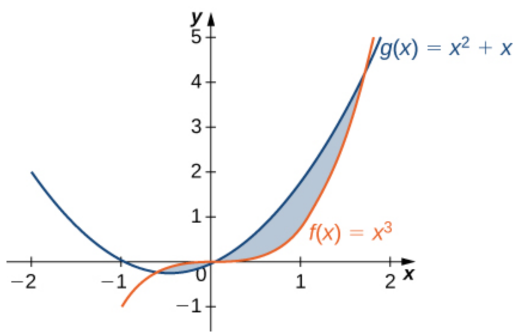


2. $y = x^2$ and $y = 3x + 4$

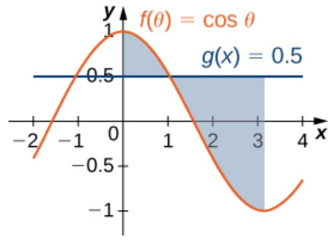


For the following exercises, split the region between the two curves into two smaller regions, then determine the area by integrating over the x-axis. Note that you will have two integrals to solve.

3. $y = x^3$ and $y = x^2 + x$

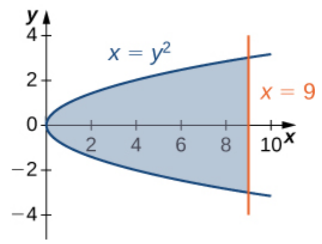


4. $y = \cos\theta$ and $y = 0.5$, for $0 \le \theta \le \pi$

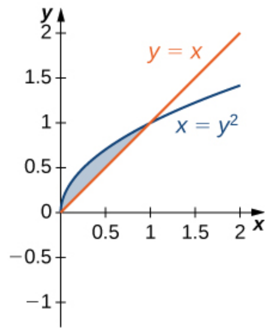


For the following exercises, determine the area of the region between the two curves by integrating over the y-axis.

5. $x = y^2$ and $x = 9$



6. $y = x$ and $x = y^2$



For the following exercises, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the x-axis.

7. $y = x^2$ and $y = -x^2 + 18x$

8. $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, and $x = 3$

9. $y = \cos x$ and $y = \cos^2 x$ on $x = [-\pi, \pi]$

10. $y = e^x$, $y = e^{2x-1}$, and $x = 0$

11. $y = e^x$, $y = e^{-x}$, $x = -1$ and $x = 1$

12. $y = e,\ y = e^x,$ and $y = e^{-x}$

13. $y = |x|$ and $y = x^2$

For the following exercises, graph the equations and shade the area of the region between the curves. If necessary, break the region into sub-regions to determine its entire area.

14. $y = \sin(\pi x),\ y = 2x,$ and $x > 0$

15. $y = 12 - x,\ y = \sqrt{x},$ and $y = 1$

16. $y = \sin x$ and $y = \cos x$ over $x = [-\pi, \pi]$

17. $y = x^3$ and $y = x^2 - 2x$ over $x = [-1, 1]$

18. $y = x^2 + 9$ and $y = 10 + 2x$ over $x = [-1, 3]$

19. $y = x^3 + 3x$ and $y = 4x$

For the following exercises, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the y-axis.

20. $x = y^3$ and $x = 3y - 2$

21. $x = 2y$ and $x = y^3 - y$

22. $x = -3 + y^2$ and $x = y - y^2$

23. $y^2 = x$ and $x = y + 2$

24. $x = |y|$ and $2x = -y^2 + 2$

25. $x = \sin y,\ x = \cos(2y),\ y = \pi/2,$ and $y = -\pi/2$

For the following exercises, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the x-axis or y-axis, whichever seems more convenient.

26. $x = y^4$ and $x = y^5$

27. $y = xe^x,\ y = e^x,\ x = 0,$ and $x = 1$

28. $y = x^6$ and $y = x^4$

29. $x = y^3 + 2y^2 + 1$ and $x = -y^2 + 1$

30. $y = |x|$ and $y = x^2 - 1$

31. $y = 4 - 3x$ and $y = \frac{1}{x}$

32. $y = \sin x,\ x = -\pi/6,\ x = \pi/6,$ and $y = \cos^3 x$

33. $y = x^2 - 3x + 2$ and $y = x^3 - 2x^2 - x + 2$

34. $y = 2\cos^3(3x),\ y = -1,\ x = \frac{\pi}{4},$ and $x = -\frac{\pi}{4}$

35. $y + y^3 = x$ and $2y = x$

36. $y = \sqrt{1 - x^2}$ and $y = x^2 - 1$

37. $y = \cos^{-1} x,\ y = \sin^{-1} x,\ x = -1,$ and $x = 1$

For the following exercises, find the exact area of the region bounded by the given equations if possible. If you are unable to determine the intersection points analytically, use a calculator to approximate the intersection points with three decimal places and determine the approximate area of the region.

38. **[T]** $x = e^y$ and $y = x - 2$

39. **[T]** $y = x^2$ and $y = \sqrt{1 - x^2}$

40. **[T]** $y = 3x^2 + 8x + 9$ and $3y = x + 24$

41. **[T]** $x = \sqrt{4 - y^2}$ and $y^2 = 1 + x^2$

42. **[T]** $x^2 = y^3$ and $x = 3y$

43. **[T]** $y = \sin^3 x + 2,\ y = \tan x,\ x = -1.5,$ and $x = 1.5$

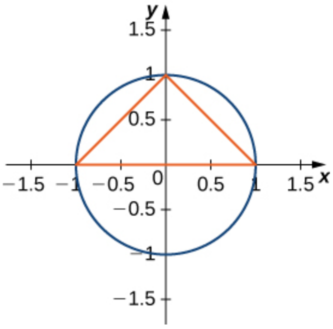
44. **[T]** $y = \sqrt{1 - x^2}$ and $y^2 = x^2$

45. **[T]** $y = \sqrt{1 - x^2}$ and $y = x^2 + 2x + 1$

46. **[T]** $x = 4 - y^2$ and $x = 1 + 3y + y^2$

47. **[T]** $y = \cos x,\ y = e^x,\ x = -\pi,$ and $x = 0$

48. The largest triangle with a base on the x-axis that fits inside the upper half of the unit circle $y^2 + x^2 = 1$ is given by $y = 1 + x$ and $y = 1 - x$. See the following figure. What is the area inside the semicircle but outside the triangle?



49. A factory selling cell phones has a marginal cost function $C(x) = 0.01x^2 - 3x + 229$, where x represents the number of cell phones, and a marginal revenue function given by $R(x) = 429 - 2x$. Find the area between the graphs of these curves and $x = 0$. What does this area represent?

50. An amusement park has a marginal cost function $C(x) = 1000e^{-x} + 5$, where x represents the number of tickets sold, and a marginal revenue function given by $R(x) = 60 - 0.1x$. Find the total profit generated when selling 550 tickets. Use a calculator to determine intersection points, if necessary, to two decimal places.

51. The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = 1 - \cos((\pi t)/2)$ whereas the speed of the tortoise is $T(t) = (1/2)\tan^{-1}(t/4)$, where t is time measured in hours and the speed is measured in miles per hour. Find the area between the curves from time $t = 0$ to the first time after one hour when the tortoise and hare are traveling at the same speed. What does it represent? Use a calculator to determine the intersection points, if necessary, accurate to three decimal places.

52. The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = (1/2) - (1/2)\cos(2\pi t)$ whereas the speed of the tortoise is $T(t) = \sqrt{t}$, where t is time measured in hours and speed is measured in kilometers per hour. If the race is over in 1 hour, who won the race and by how much? Use a calculator to determine the intersection points, if necessary, accurate to three decimal places.

For the following exercises, find the area between the curves by integrating with respect to x and then with respect to y. Is one method easier than the other? Do you obtain the same answer?

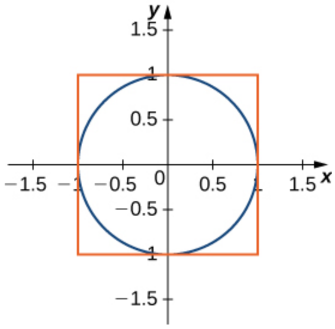
53. $y = x^2 + 2x + 1$ and $y = -x^2 - 3x + 4$

54. $y = x^4$ and $x = y^5$

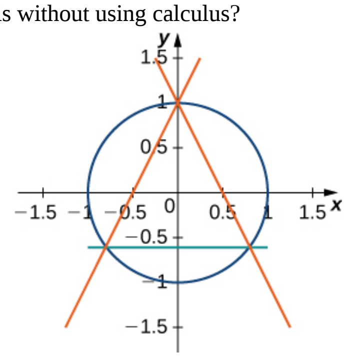
55. $x = y^2 - 2$ and $x = 2y$

For the following exercises, solve using calculus, then check your answer with geometry.

56. Determine the equations for the sides of the square that touches the unit circle on all four sides, as seen in the following figure. Find the area between the perimeter of this square and the unit circle. Is there another way to solve this without using calculus?



57. Find the area between the perimeter of the unit circle and the triangle created from $y = 2x + 1$, $y = 1 - 2x$ and $y = -\frac{3}{5}$, as seen in the following figure. Is there a way to solve this without using calculus?



6.2 | Determining Volumes by Slicing

Learning Objectives

6.2.1 Determine the volume of a solid by integrating a cross-section (the slicing method).
6.2.2 Find the volume of a solid of revolution using the disk method.
6.2.3 Find the volume of a solid of revolution with a cavity using the washer method.

In the preceding section, we used definite integrals to find the area between two curves. In this section, we use definite integrals to find volumes of three-dimensional solids. We consider three approaches—slicing, disks, and washers—for finding these volumes, depending on the characteristics of the solid.

Volume and the Slicing Method

Just as area is the numerical measure of a two-dimensional region, volume is the numerical measure of a three-dimensional solid. Most of us have computed volumes of solids by using basic geometric formulas. The volume of a rectangular solid, for example, can be computed by multiplying length, width, and height: $V = lwh.$ The formulas for the volume of a sphere $\left(V = \frac{4}{3}\pi r^3\right),$ a cone $\left(V = \frac{1}{3}\pi r^2 h\right),$ and a pyramid $\left(V = \frac{1}{3}Ah\right)$ have also been introduced. Although some of these formulas were derived using geometry alone, all these formulas can be obtained by using integration.

We can also calculate the volume of a cylinder. Although most of us think of a cylinder as having a circular base, such as a soup can or a metal rod, in mathematics the word *cylinder* has a more general meaning. To discuss cylinders in this more general context, we first need to define some vocabulary.

We define the **cross-section** of a solid to be the intersection of a plane with the solid. A *cylinder* is defined as any solid that can be generated by translating a plane region along a line perpendicular to the region, called the *axis* of the cylinder. Thus, all cross-sections perpendicular to the axis of a cylinder are identical. The solid shown in Figure 6.11 is an example of a cylinder with a noncircular base. To calculate the volume of a cylinder, then, we simply multiply the area of the cross-section by the height of the cylinder: $V = A \cdot h.$ In the case of a right circular cylinder (soup can), this becomes $V = \pi r^2 h.$

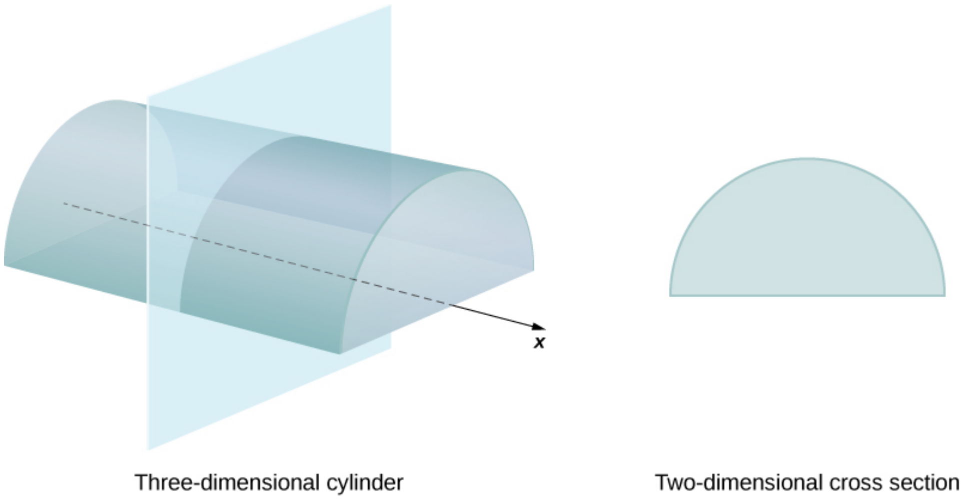


Figure 6.11 Each cross-section of a particular cylinder is identical to the others.

If a solid does not have a constant cross-section (and it is not one of the other basic solids), we may not have a formula for its volume. In this case, we can use a definite integral to calculate the volume of the solid. We do this by slicing the solid into pieces, estimating the volume of each slice, and then adding those estimated volumes together. The slices should all be parallel to one another, and when we put all the slices together, we should get the whole solid. Consider, for example, the solid S shown in Figure 6.12, extending along the x-axis.

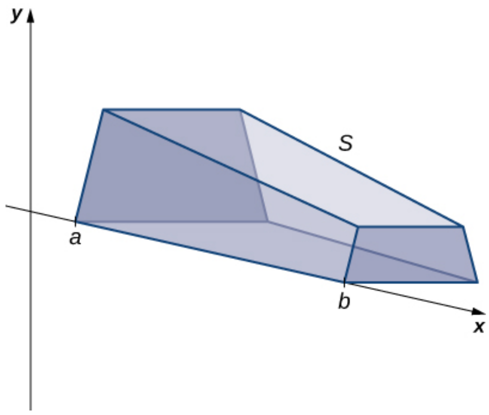


Figure 6.12 A solid with a varying cross-section.

We want to divide S into slices perpendicular to the x-axis. As we see later in the chapter, there may be times when we want to slice the solid in some other direction—say, with slices perpendicular to the y-axis. The decision of which way to slice the solid is very important. If we make the wrong choice, the computations can get quite messy. Later in the chapter, we examine some of these situations in detail and look at how to decide which way to slice the solid. For the purposes of this section, however, we use slices perpendicular to the x-axis.

Because the cross-sectional area is not constant, we let $A(x)$ represent the area of the cross-section at point x. Now let $P = \{x_0, x_1 \ldots, X_n\}$ be a regular partition of $[a, b]$, and for $i = 1, 2, \ldots n$, let S_i represent the slice of S stretching from x_{i-1} to x_i. The following figure shows the sliced solid with $n = 3$.

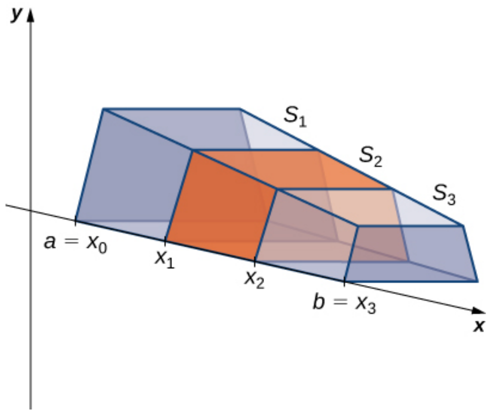


Figure 6.13 The solid S has been divided into three slices perpendicular to the x-axis.

Finally, for $i = 1, 2, \ldots n$, let x_i^* be an arbitrary point in $[x_{i-1}, x_i]$. Then the volume of slice S_i can be estimated by $V(S_i) \approx A(x_i^*)\Delta x$. Adding these approximations together, we see the volume of the entire solid S can be approximated by

$$V(S) \approx \sum_{i=1}^{n} A(x_i^*)\Delta x.$$

By now, we can recognize this as a Riemann sum, and our next step is to take the limit as $n \to \infty$. Then we have

$$V(S) = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)dx.$$

The technique we have just described is called the **slicing method**. To apply it, we use the following strategy.

Problem-Solving Strategy: Finding Volumes by the Slicing Method

1. Examine the solid and determine the shape of a cross-section of the solid. It is often helpful to draw a picture if one is not provided.
2. Determine a formula for the area of the cross-section.
3. Integrate the area formula over the appropriate interval to get the volume.

Recall that in this section, we assume the slices are perpendicular to the x-axis. Therefore, the area formula is in terms of x and the limits of integration lie on the x-axis. However, the problem-solving strategy shown here is valid regardless of how we choose to slice the solid.

Example 6.6

Deriving the Formula for the Volume of a Pyramid

We know from geometry that the formula for the volume of a pyramid is $V = \frac{1}{3}Ah$. If the pyramid has a square base, this becomes $V = \frac{1}{3}a^2h$, where a denotes the length of one side of the base. We are going to use the slicing method to derive this formula.

Solution

We want to apply the slicing method to a pyramid with a square base. To set up the integral, consider the pyramid shown in **Figure 6.14**, oriented along the x-axis.

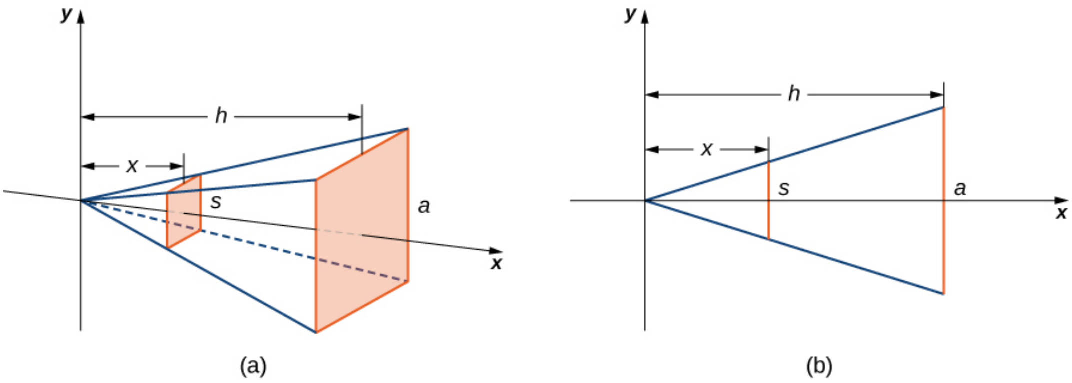


Figure 6.14 (a) A pyramid with a square base is oriented along the x-axis. (b) A two-dimensional view of the pyramid is seen from the side.

We first want to determine the shape of a cross-section of the pyramid. We know the base is a square, so the cross-sections are squares as well (step 1). Now we want to determine a formula for the area of one of these cross-sectional squares. Looking at **Figure 6.14**(b), and using a proportion, since these are similar triangles, we have

$$\frac{s}{a} = \frac{x}{h} \text{ or } s = \frac{ax}{h}.$$

Therefore, the area of one of the cross-sectional squares is

$$A(x) = s^2 = \left(\frac{ax}{h}\right)^2 \text{(step 2)}.$$

Then we find the volume of the pyramid by integrating from 0 to h (step 3):

$$\begin{aligned} V &= \int_0^h A(x)dx \\ &= \int_0^h \left(\frac{ax}{h}\right)^2 dx = \frac{a^2}{h^2}\int_0^h x^2 dx \\ &= \left[\frac{a^2}{h^2}\left(\frac{1}{3}x^3\right)\right]\Bigg|_0^h = \frac{1}{3}a^2 h. \end{aligned}$$

This is the formula we were looking for.

6.6 Use the slicing method to derive the formula $V = \frac{1}{3}\pi r^2 h$ for the volume of a circular cone.

Solids of Revolution

If a region in a plane is revolved around a line in that plane, the resulting solid is called a **solid of revolution**, as shown in the following figure.

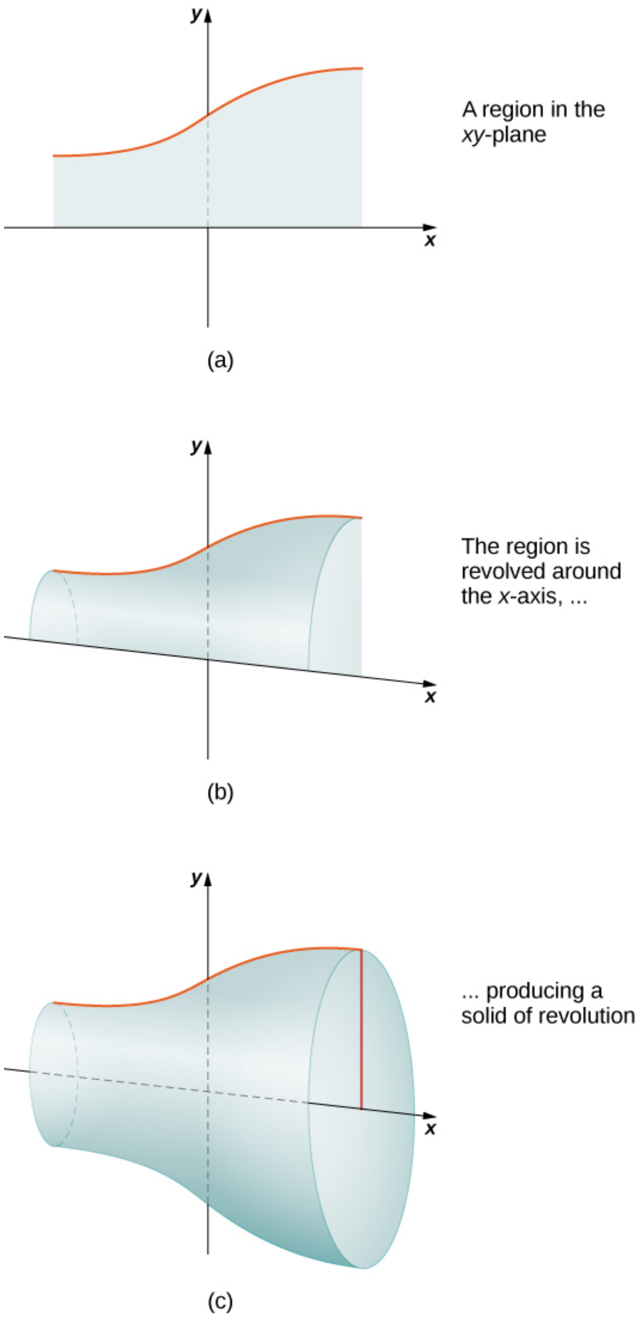


Figure 6.15 (a) This is the region that is revolved around the x-axis. (b) As the region begins to revolve around the axis, it sweeps out a solid of revolution. (c) This is the solid that results when the revolution is complete.

Solids of revolution are common in mechanical applications, such as machine parts produced by a lathe. We spend the rest of this section looking at solids of this type. The next example uses the slicing method to calculate the volume of a solid of revolution.

Use an online **integral calculator (http://www.openstax.org/l/20_IntCalc2)** to learn more.

Example 6.7

Using the Slicing Method to find the Volume of a Solid of Revolution

Use the slicing method to find the volume of the solid of revolution bounded by the graphs of $f(x) = x^2 - 4x + 5,\ x = 1,$ and $x = 4,$ and rotated about the x-axis.

Solution

Using the problem-solving strategy, we first sketch the graph of the quadratic function over the interval $[1, 4]$ as shown in the following figure.

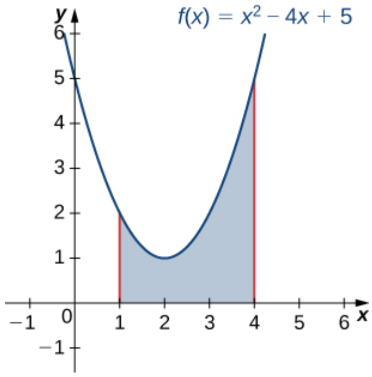


Figure 6.16 A region used to produce a solid of revolution.

Next, revolve the region around the x-axis, as shown in the following figure.

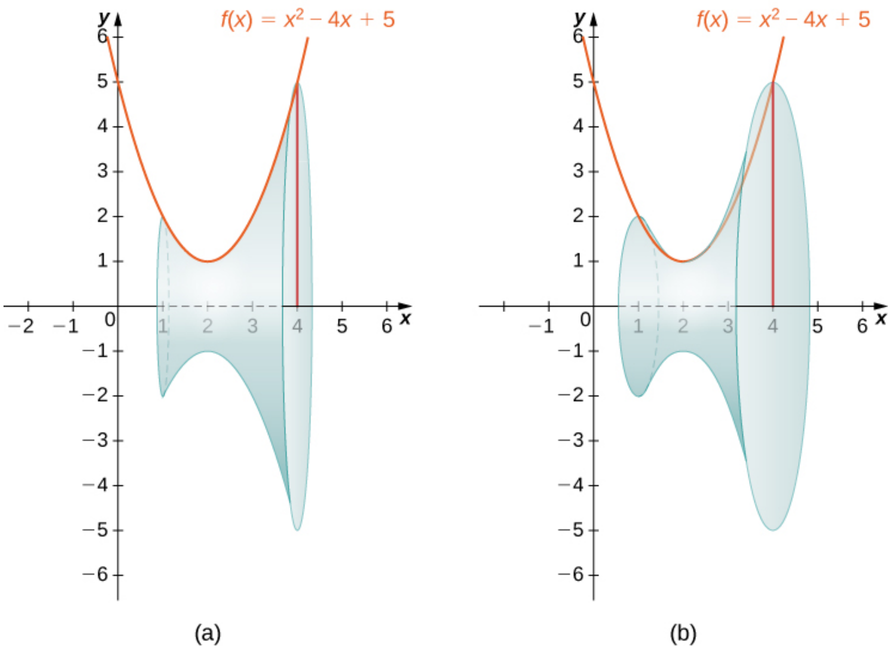


Figure 6.17 Two views, (a) and (b), of the solid of revolution produced by revolving the region in **Figure 6.16** about the x-axis.

Since the solid was formed by revolving the region around the x-axis, the cross-sections are circles (step 1). The area of the cross-section, then, is the area of a circle, and the radius of the circle is given by $f(x)$. Use the formula for the area of the circle:

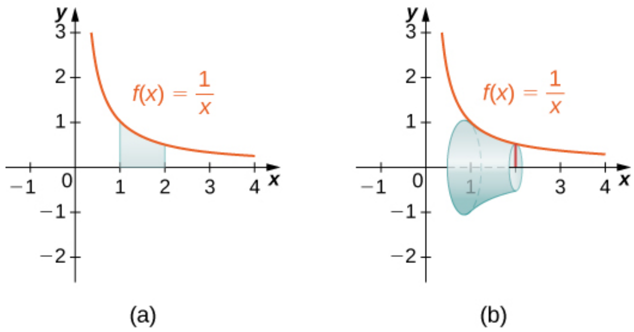
$$A(x) = \pi r^2 = \pi[f(x)]^2 = \pi\left(x^2 - 4x + 5\right)^2 \text{ (step 2).}$$

The volume, then, is (step 3)

$$\begin{aligned} V &= \int_a^b A(x)dx \\ &= \int_1^4 \pi\left(x^2 - 4x + 5\right)^2 dx = \pi \int_1^4 \left(x^4 - 8x^3 + 26x^2 - 40x + 25\right)dx \\ &= \pi\left(\frac{x^5}{5} - 2x^4 + \frac{26x^3}{3} - 20x^2 + 25x\right)\Bigg|_1^4 = \frac{78}{5}\pi. \end{aligned}$$

The volume is $78\pi/5$.

6.7 Use the method of slicing to find the volume of the solid of revolution formed by revolving the region between the graph of the function $f(x) = 1/x$ and the x-axis over the interval $[1, 2]$ around the x-axis. See the following figure.



The Disk Method

When we use the slicing method with solids of revolution, it is often called the **disk method** because, for solids of revolution, the slices used to over approximate the volume of the solid are disks. To see this, consider the solid of revolution generated by revolving the region between the graph of the function $f(x) = (x - 1)^2 + 1$ and the x-axis over the interval $[-1, 3]$ around the x-axis. The graph of the function and a representative disk are shown in **Figure 6.18**(a) and (b). The region of revolution and the resulting solid are shown in **Figure 6.18**(c) and (d).

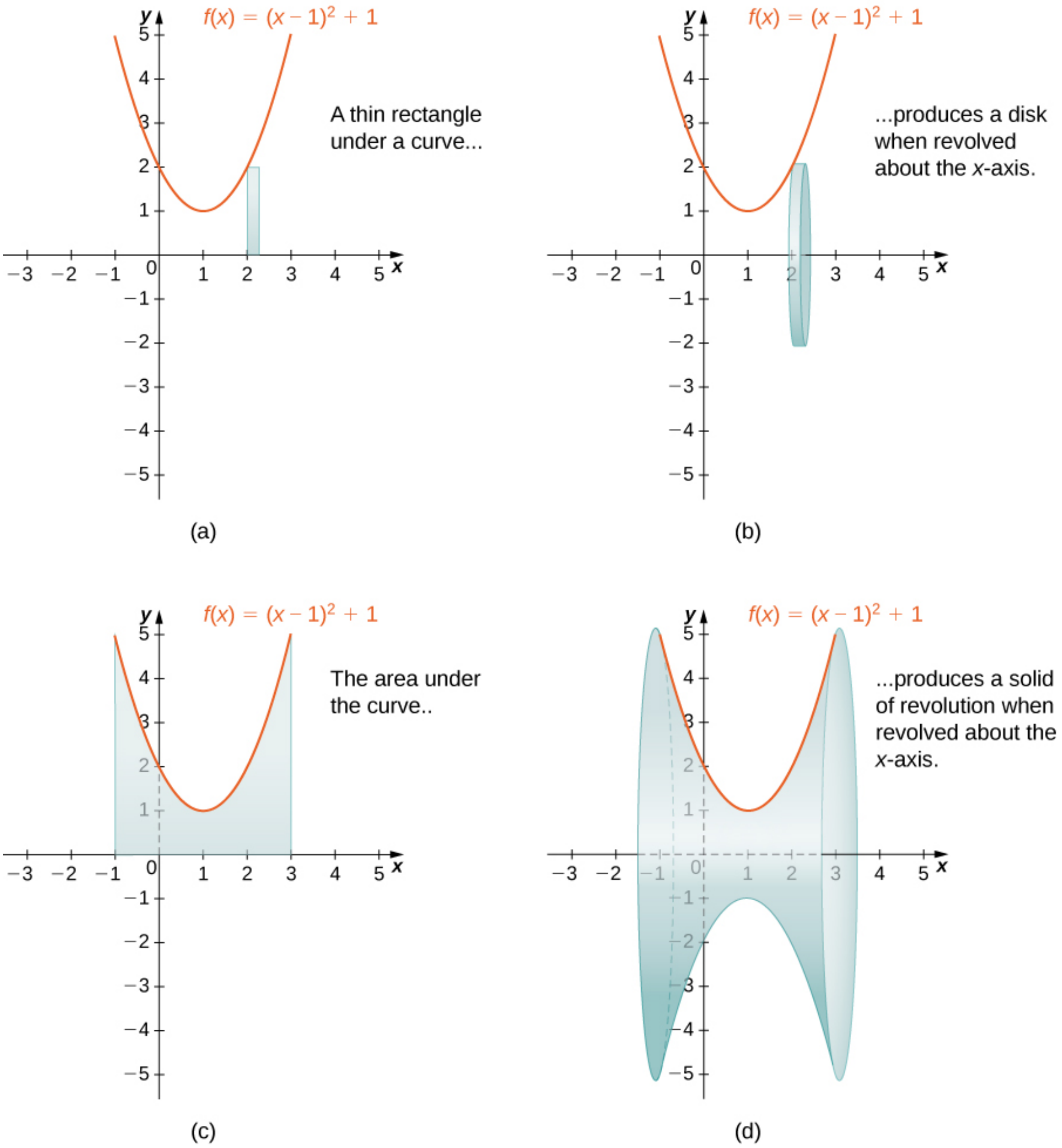


Figure 6.18 (a) A thin rectangle for approximating the area under a curve. (b) A representative disk formed by revolving the rectangle about the x-axis. (c) The region under the curve is revolved about the x-axis, resulting in (d) the solid of revolution.

We already used the formal Riemann sum development of the volume formula when we developed the slicing method. We know that

$$V = \int_a^b A(x)dx.$$

The only difference with the disk method is that we know the formula for the cross-sectional area ahead of time; it is the area of a circle. This gives the following rule.

Rule: The Disk Method

Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of $f(x),$ below by the

x-axis, on the left by the line $x = a,$ and on the right by the line $x = b.$ Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by

$$V = \int_a^b \pi [f(x)]^2 dx. \quad (6.3)$$

The volume of the solid we have been studying (Figure 6.18) is given by

$$\begin{aligned} V &= \int_a^b \pi [f(x)]^2 dx \\ &= \int_{-1}^{3} \pi \left[(x-1)^2 + 1\right]^2 dx = \pi \int_{-1}^{3} \left[(x-1)^4 + 2(x-1)^2 + 1\right] dx \\ &= \pi \left[\frac{1}{5}(x-1)^5 + \frac{2}{3}(x-1)^3 + x\right]\Bigg|_{-1}^{3} = \pi \left[\left(\frac{32}{5} + \frac{16}{3} + 3\right) - \left(-\frac{32}{5} - \frac{16}{3} - 1\right)\right] = \frac{412\pi}{15} \text{ units}^3. \end{aligned}$$

Let's look at some examples.

Example 6.8

Using the Disk Method to Find the Volume of a Solid of Revolution 1

Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x) = \sqrt{x}$ and the x-axis over the interval $[1, 4]$ around the x-axis.

Solution

The graphs of the function and the solid of revolution are shown in the following figure.

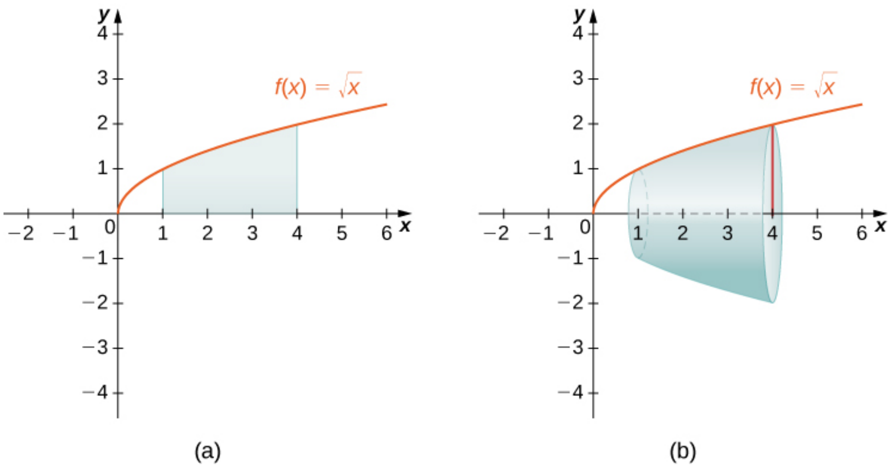


Figure 6.19 (a) The function $f(x) = \sqrt{x}$ over the interval $[1, 4]$. (b) The solid of revolution obtained by revolving the region under the graph of $f(x)$ about the x-axis.

We have

$$\begin{aligned} V &= \int_a^b \pi[f(x)]^2\,dx \\ &= \int_1^4 \pi[\sqrt{x}]^2 dx = \pi\int_1^4 x\,dx \\ &= \frac{\pi}{2}x^2\Big|_1^4 = \frac{15\pi}{2}. \end{aligned}$$

The volume is $(15\pi)/2$ units^3.

6.8 Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x) = \sqrt{4 - x}$ and the x-axis over the interval $[0, 4]$ around the x-axis.

So far, our examples have all concerned regions revolved around the x-axis, but we can generate a solid of revolution by revolving a plane region around any horizontal or vertical line. In the next example, we look at a solid of revolution that has been generated by revolving a region around the y-axis. The mechanics of the disk method are nearly the same as when the x-axis is the axis of revolution, but we express the function in terms of y and we integrate with respect to y as well. This is summarized in the following rule.

Rule: The Disk Method for Solids of Revolution around the *y*-axis

Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y-axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the y-axis is given by

$$V = \int_c^d \pi[g(y)]^2\,dy. \qquad (6.4)$$

The next example shows how this rule works in practice.

Example 6.9

Using the Disk Method to Find the Volume of a Solid of Revolution 2

Let R be the region bounded by the graph of $g(y) = \sqrt{4 - y}$ and the y-axis over the y-axis interval $[0, 4]$. Use the disk method to find the volume of the solid of revolution generated by rotating R around the y-axis.

Solution

Figure 6.20 shows the function and a representative disk that can be used to estimate the volume. Notice that since we are revolving the function around the y-axis, the disks are horizontal, rather than vertical.

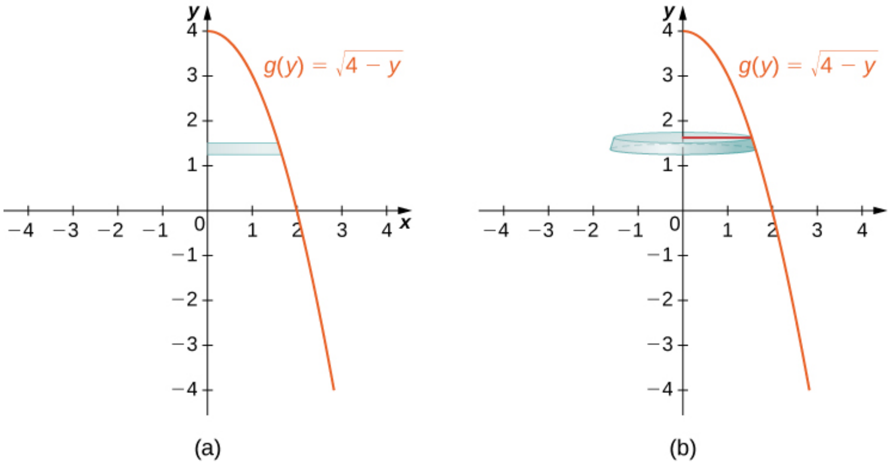


Figure 6.20 (a) Shown is a thin rectangle between the curve of the function $g(y) = \sqrt{4 - y}$ and the y-axis. (b) The rectangle forms a representative disk after revolution around the y-axis.

The region to be revolved and the full solid of revolution are depicted in the following figure.

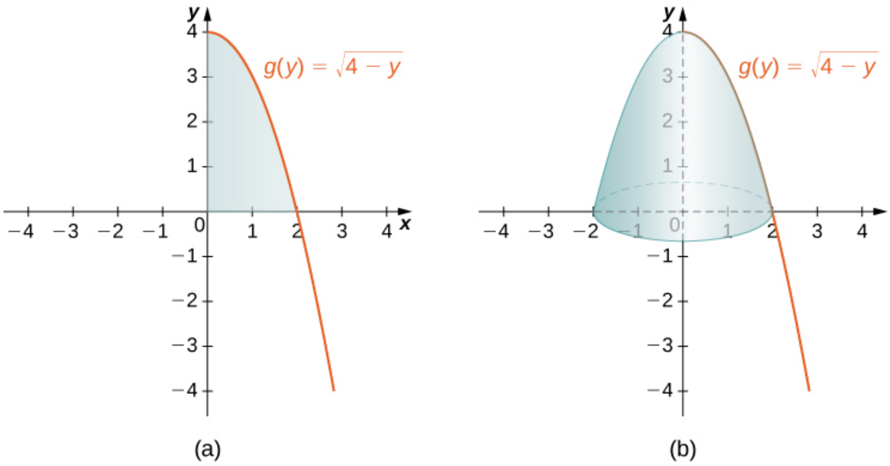


Figure 6.21 (a) The region to the left of the function $g(y) = \sqrt{4 - y}$ over the y-axis interval $[0, 4]$. (b) The solid of revolution formed by revolving the region about the y-axis.

To find the volume, we integrate with respect to y. We obtain

$$\begin{aligned} V &= \int_c^d \pi[g(y)]^2\,dy \\ &= \int_0^4 \pi[\sqrt{4 - y}]^2\,dy = \pi\int_0^4 (4 - y)dy \\ &= \pi\left[4y - \frac{y^2}{2}\right]\Bigg|_0^4 = 8\pi. \end{aligned}$$

The volume is 8π units^3.

6.9 Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $g(y) = y$ and the y-axis over the interval $[1, 4]$ around the y-axis.

The Washer Method

Some solids of revolution have cavities in the middle; they are not solid all the way to the axis of revolution. Sometimes, this is just a result of the way the region of revolution is shaped with respect to the axis of revolution. In other cases, cavities arise when the region of revolution is defined as the region between the graphs of two functions. A third way this can happen is when an axis of revolution other than the x-axis or y-axis is selected.

When the solid of revolution has a cavity in the middle, the slices used to approximate the volume are not disks, but washers (disks with holes in the center). For example, consider the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1$ over the interval $[1, 4]$. When this region is revolved around the x-axis, the result is a solid with a cavity in the middle, and the slices are washers. The graph of the function and a representative washer are shown in **Figure 6.22**(a) and (b). The region of revolution and the resulting solid are shown in **Figure 6.22**(c) and (d).

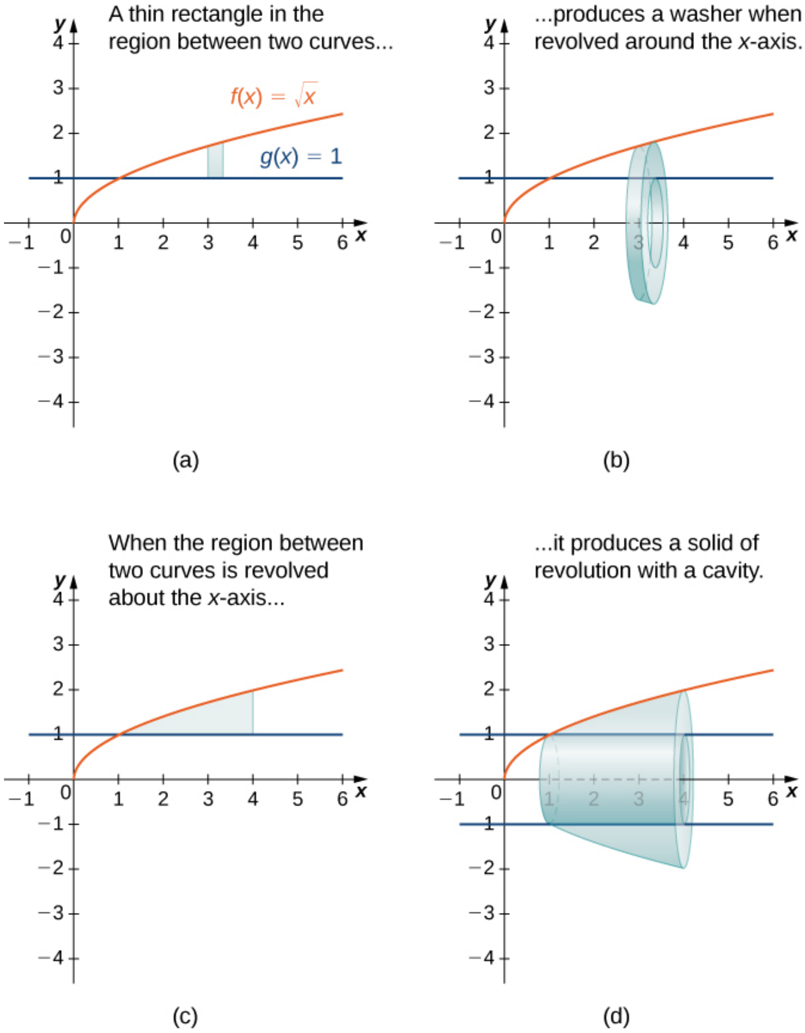


Figure 6.22 (a) A thin rectangle in the region between two curves. (b) A representative disk formed by revolving the rectangle about the x-axis. (c) The region between the curves over the given interval. (d) The resulting solid of revolution.

The cross-sectional area, then, is the area of the outer circle less the area of the inner circle. In this case,

$$A(x) = \pi(\sqrt{x})^2 - \pi(1)^2 = \pi(x - 1).$$

Then the volume of the solid is

$$\begin{aligned} V &= \int_a^b A(x)dx \\ &= \int_1^4 \pi(x-1)dx = \pi\left[\frac{x^2}{2} - x\right]\Bigg|_1^4 = \frac{9}{2}\pi \text{ units}^3. \end{aligned}$$

Generalizing this process gives the **washer method**.

Rule: The Washer Method

Suppose $f(x)$ and $g(x)$ are continuous, nonnegative functions such that $f(x) \geq g(x)$ over $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving R around the x-axis is given by

$$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx. \tag{6.5}$$

Example 6.10

Using the Washer Method

Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = 1/x$ over the interval $[1, 4]$ around the x-axis.

Solution

The graphs of the functions and the solid of revolution are shown in the following figure.

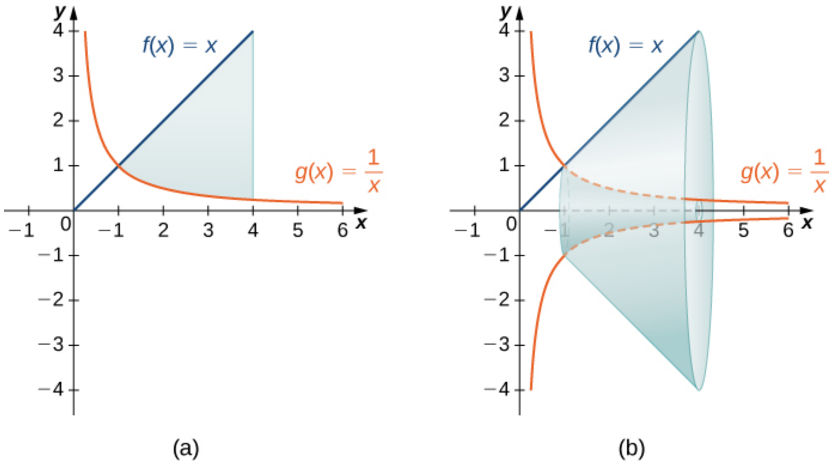


Figure 6.23 (a) The region between the graphs of the functions $f(x) = x$ and $g(x) = 1/x$ over the interval $[1, 4]$. (b) Revolving the region about the x-axis generates a solid of revolution with a cavity in the middle.

We have

$$\begin{aligned} V &= \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx \\ &= \pi\int_1^4 \left[x^2 - \left(\frac{1}{x}\right)^2\right]dx = \pi\left[\frac{x^3}{3} + \frac{1}{x}\right]\Bigg|_1^4 = \frac{81\pi}{4}\text{ units}^3. \end{aligned}$$

6.10 Find the volume of a solid of revolution formed by revolving the region bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = 1/x$ over the interval $[1, 3]$ around the x-axis.

As with the disk method, we can also apply the washer method to solids of revolution that result from revolving a region around the y-axis. In this case, the following rule applies.

Rule: The Washer Method for Solids of Revolution around the *y*-axis

Suppose $u(y)$ and $v(y)$ are continuous, nonnegative functions such that $v(y) \le u(y)$ for $y \in [c, d]$. Let Q denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the y-axis is given by

$$V = \int_c^d \pi\left[(u(y))^2 - (v(y))^2\right]dy.$$

Rather than looking at an example of the washer method with the y-axis as the axis of revolution, we now consider an example in which the axis of revolution is a line other than one of the two coordinate axes. The same general method applies, but you may have to visualize just how to describe the cross-sectional area of the volume.

Example 6.11

The Washer Method with a Different Axis of Revolution

Find the volume of a solid of revolution formed by revolving the region bounded above by $f(x) = 4 - x$ and below by the x-axis over the interval $[0, 4]$ around the line $y = -2$.

Solution

The graph of the region and the solid of revolution are shown in the following figure.

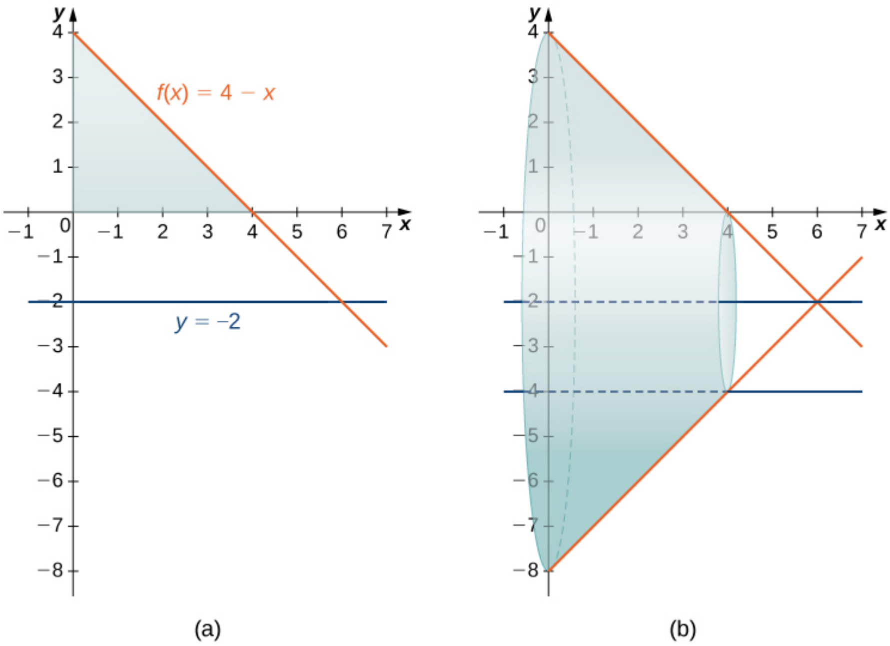


Figure 6.24 (a) The region between the graph of the function $f(x) = 4 - x$ and the x-axis over the interval $[0, 4]$. (b) Revolving the region about the line $y = -2$ generates a solid of revolution with a cylindrical hole through its middle.

We can't apply the volume formula to this problem directly because the axis of revolution is not one of the coordinate axes. However, we still know that the area of the cross-section is the area of the outer circle less the area of the inner circle. Looking at the graph of the function, we see the radius of the outer circle is given by $f(x) + 2,$ which simplifies to

$$f(x) + 2 = (4 - x) + 2 = 6 - x.$$

The radius of the inner circle is $g(x) = 2.$ Therefore, we have

$$\begin{aligned} V &= \int_0^4 \pi\left[(6 - x)^2 - (2)^2\right]dx \\ &= \pi\int_0^4 \left(x^2 - 12x + 32\right)dx \;= \pi\left[\frac{x^3}{3} - 6x^2 + 32x\right]\Bigg|_0^4 = \frac{160\pi}{3}\text{ units}^3. \end{aligned}$$

6.11 Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x + 2$ and below by the x-axis over the interval $[0,\ 3]$ around the line $y = -1.$

6.2 EXERCISES

58. Derive the formula for the volume of a sphere using the slicing method.

59. Use the slicing method to derive the formula for the volume of a cone.

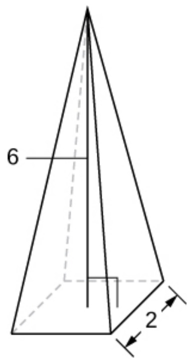
60. Use the slicing method to derive the formula for the volume of a tetrahedron with side length a.

61. Use the disk method to derive the formula for the volume of a trapezoidal cylinder.

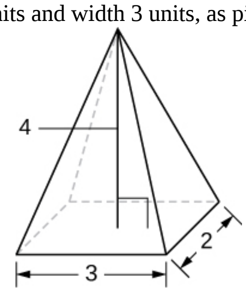
62. Explain when you would use the disk method versus the washer method. When are they interchangeable?

For the following exercises, draw a typical slice and find the volume using the slicing method for the given volume.

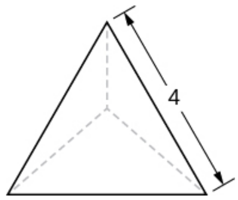
63. A pyramid with height 6 units and square base of side 2 units, as pictured here.



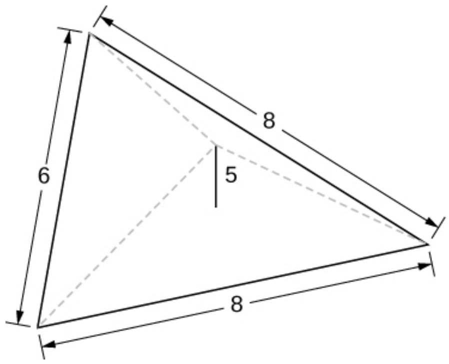
64. A pyramid with height 4 units and a rectangular base with length 2 units and width 3 units, as pictured here.



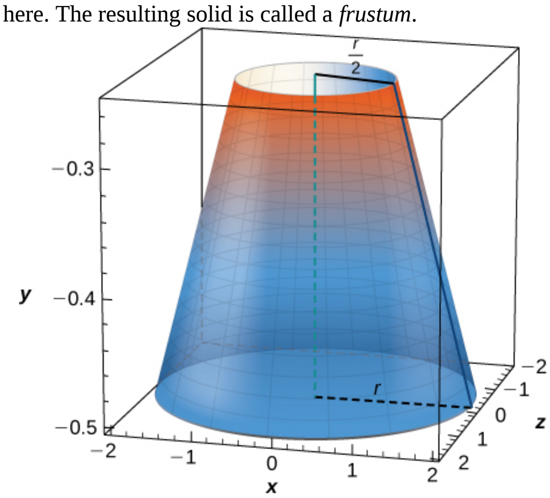
65. A tetrahedron with a base side of 4 units, as seen here.



66. A pyramid with height 5 units, and an isosceles triangular base with lengths of 6 units and 8 units, as seen here.



67. A cone of radius r and height h has a smaller cone of radius $r/2$ and height $h/2$ removed from the top, as seen here. The resulting solid is called a *frustum*.



For the following exercises, draw an outline of the solid and find the volume using the slicing method.

68. The base is a circle of radius a. The slices perpendicular to the base are squares.

69. The base is a triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Slices perpendicular to the x-axis are semicircles.

70. The base is the region under the parabola $y = 1 - x^2$ in the first quadrant. Slices perpendicular to the xy-plane are squares.

71. The base is the region under the parabola $y = 1 - x^2$ and above the x-axis. Slices perpendicular to the y-axis are squares.

72. The base is the region enclosed by $y = x^2$ and $y = 9$. Slices perpendicular to the x-axis are right isosceles triangles. The intersection of one of these slices and the base is the leg of the triangle.

73. The base is the area between $y = x$ and $y = x^2$. Slices perpendicular to the x-axis are semicircles.

For the following exercises, draw the region bounded by the curves. Then, use the disk method to find the volume when the region is rotated around the x-axis.

74. $x + y = 8,\ x = 0,$ and $y = 0$

75. $y = 2x^2,\ x = 0,\ x = 4,$ and $y = 0$

76. $y = e^x + 1,\ x = 0,\ x = 1,$ and $y = 0$

77. $y = x^4,\ x = 0,$ and $y = 1$

78. $y = \sqrt{x},\ x = 0,\ x = 4,$ and $y = 0$

79. $y = \sin x,\ y = \cos x,$ and $x = 0$

80. $y = \frac{1}{x},\ x = 2,$ and $y = 3$

81. $x^2 - y^2 = 9$ and $x + y = 9,\ y = 0$ and $x = 0$

For the following exercises, draw the region bounded by the curves. Then, find the volume when the region is rotated around the y-axis.

82. $y = 4 - \frac{1}{2}x,\ x = 0,$ and $y = 0$

83. $y = 2x^3,\ x = 0,\ x = 1,$ and $y = 0$

84. $y = 3x^2,\ x = 0,$ and $y = 3$

85. $y = \sqrt{4 - x^2},\ y = 0,$ and $x = 0$

86. $y = \frac{1}{\sqrt{x + 1}},\ x = 0,$ and $x = 3$

87. $x = \sec(y)$ and $y = \frac{\pi}{4},\ y = 0$ and $x = 0$

88. $y = \frac{1}{x + 1},\ x = 0,$ and $x = 2$

89. $y = 4 - x,\ y = x,$ and $x = 0$

For the following exercises, draw the region bounded by the curves. Then, find the volume when the region is rotated around the x-axis.

90. $y = x + 2,\ y = x + 6,\ x = 0,$ and $x = 5$

91. $y = x^2$ and $y = x + 2$

92. $x^2 = y^3$ and $x^3 = y^2$

93. $y = 4 - x^2$ and $y = 2 - x$

94. **[T]** $y = \cos x,\ y = e^{-x},\ x = 0,$ and $x = 1.2927$

95. $y = \sqrt{x}$ and $y = x^2$

96. $y = \sin x,\ y = 5\sin x,\ x = 0$ and $x = \pi$

97. $y = \sqrt{1 + x^2}$ and $y = \sqrt{4 - x^2}$

For the following exercises, draw the region bounded by the curves. Then, use the washer method to find the volume when the region is revolved around the y-axis.

98. $y = \sqrt{x},\ x = 4,$ and $y = 0$

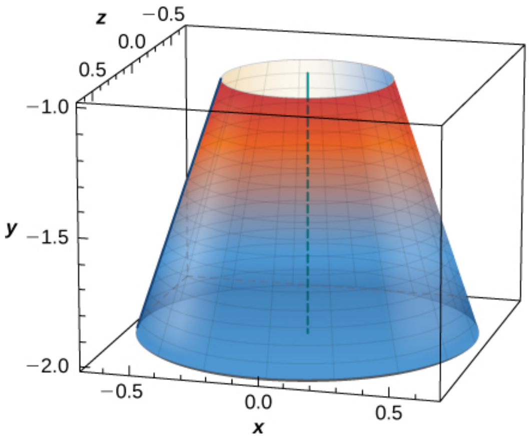
99. $y = x + 2,\ y = 2x - 1,$ and $x = 0$

100. $y = \sqrt[3]{x}$ and $y = x^3$

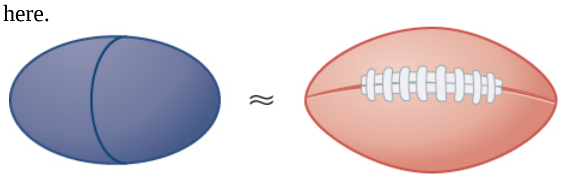
101. $x = e^{2y},\ x = y^2,\ y = 0,$ and $y = \ln(2)$

102. $x = \sqrt{9 - y^2},\ x = e^{-y},\ y = 0,$ and $y = 3$

103. Yogurt containers can be shaped like frustums. Rotate the line $y = \frac{1}{m}x$ around the y-axis to find the volume between $y = a$ and $y = b$.

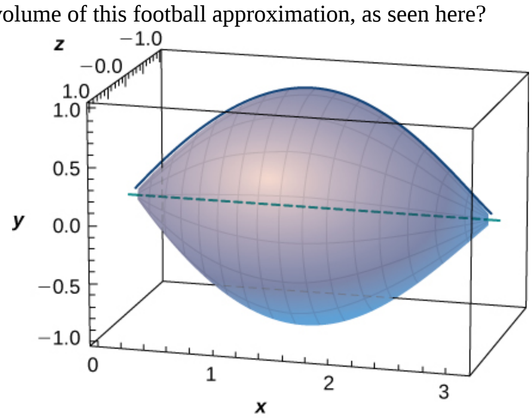


104. Rotate the ellipse $\left(x^2/a^2\right) + \left(y^2/b^2\right) = 1$ around the x-axis to approximate the volume of a football, as seen here.

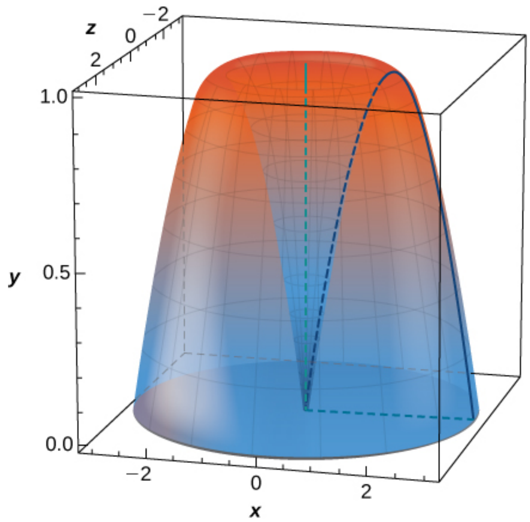


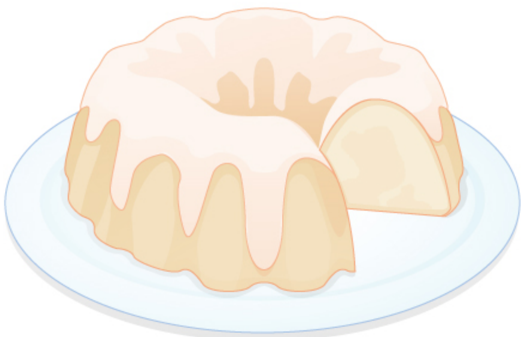
105. Rotate the ellipse $\left(x^2/a^2\right) + \left(y^2/b^2\right) = 1$ around the y-axis to approximate the volume of a football.

106. A better approximation of the volume of a football is given by the solid that comes from rotating $y = \sin x$ around the x-axis from $x = 0$ to $x = \pi$. What is the volume of this football approximation, as seen here?



107. What is the volume of the Bundt cake that comes from rotating $y = \sin x$ around the y-axis from $x = 0$ to $x = \pi$?



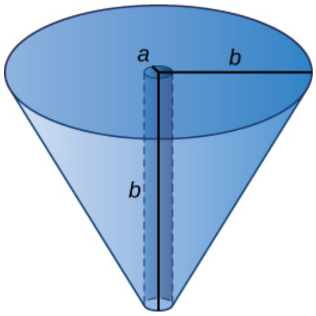


For the following exercises, find the volume of the solid described.

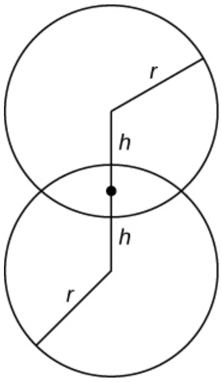
108. The base is the region between $y = x$ and $y = x^2$. Slices perpendicular to the x-axis are semicircles.

109. The base is the region enclosed by the generic ellipse $\left(x^2/a^2\right) + \left(y^2/b^2\right) = 1$. Slices perpendicular to the x-axis are semicircles.

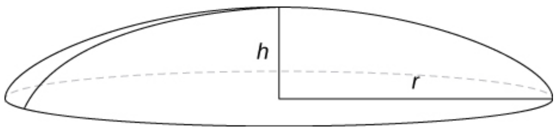
110. Bore a hole of radius a down the axis of a right cone and through the base of radius b, as seen here.



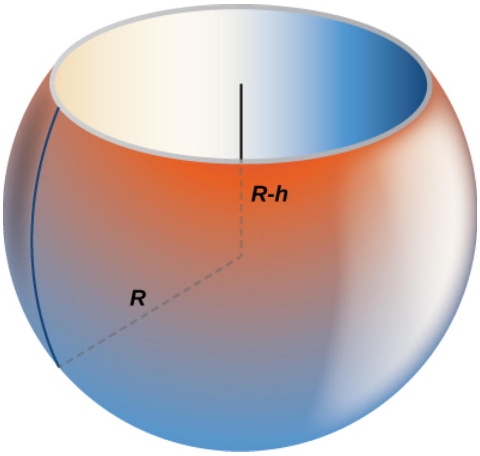
111. Find the volume common to two spheres of radius r with centers that are $2h$ apart, as shown here.



112. Find the volume of a spherical cap of height h and radius r where $h < r$, as seen here.



113. Find the volume of a sphere of radius R with a cap of height h removed from the top, as seen here.



6.3 | Volumes of Revolution: Cylindrical Shells

Learning Objectives

6.3.1 Calculate the volume of a solid of revolution by using the method of cylindrical shells.
6.3.2 Compare the different methods for calculating a volume of revolution.

In this section, we examine the method of cylindrical shells, the final method for finding the volume of a solid of revolution. We can use this method on the same kinds of solids as the disk method or the washer method; however, with the disk and washer methods, we integrate along the coordinate axis parallel to the axis of revolution. With the method of cylindrical shells, we integrate along the coordinate axis *perpendicular* to the axis of revolution. The ability to choose which variable of integration we want to use can be a significant advantage with more complicated functions. Also, the specific geometry of the solid sometimes makes the method of using cylindrical shells more appealing than using the washer method. In the last part of this section, we review all the methods for finding volume that we have studied and lay out some guidelines to help you determine which method to use in a given situation.

The Method of Cylindrical Shells

Again, we are working with a solid of revolution. As before, we define a region R, bounded above by the graph of a function $y = f(x)$, below by the x-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in **Figure 6.25**(a). We then revolve this region around the y-axis, as shown in **Figure 6.25**(b). Note that this is different from what we have done before. Previously, regions defined in terms of functions of x were revolved around the x-axis or a line parallel to it.

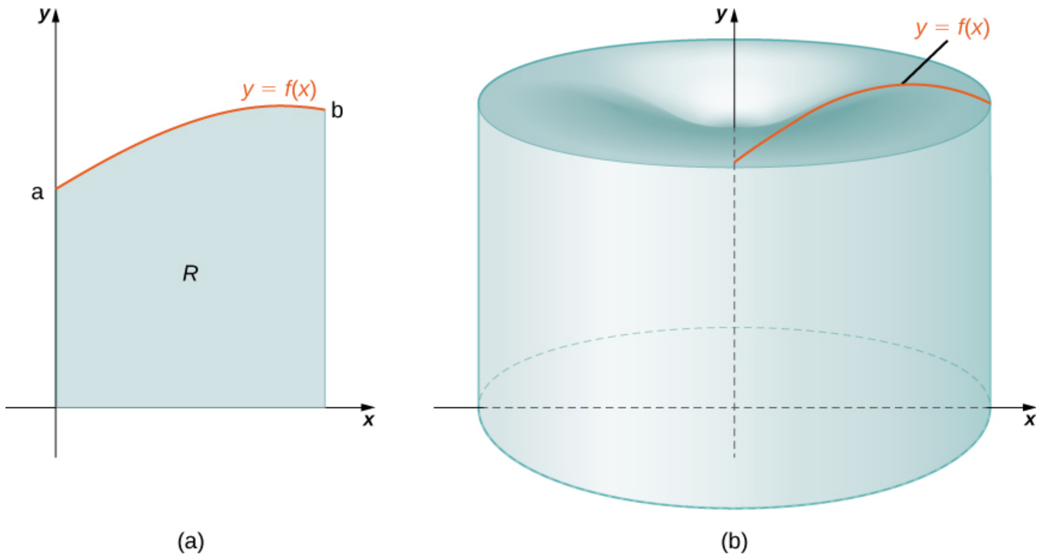


Figure 6.25 (a) A region bounded by the graph of a function of x. (b) The solid of revolution formed when the region is revolved around the y-axis.

As we have done many times before, partition the interval $[a, b]$ using a regular partition, $P = \{x_0, x_1, \ldots, x_n\}$ and, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$. Then, construct a rectangle over the interval $[x_{i-1}, x_i]$ of height $f(x_i^*)$ and width Δx. A representative rectangle is shown in **Figure 6.26**(a). When that rectangle is revolved around the y-axis, instead of a disk or a washer, we get a cylindrical shell, as shown in the following figure.

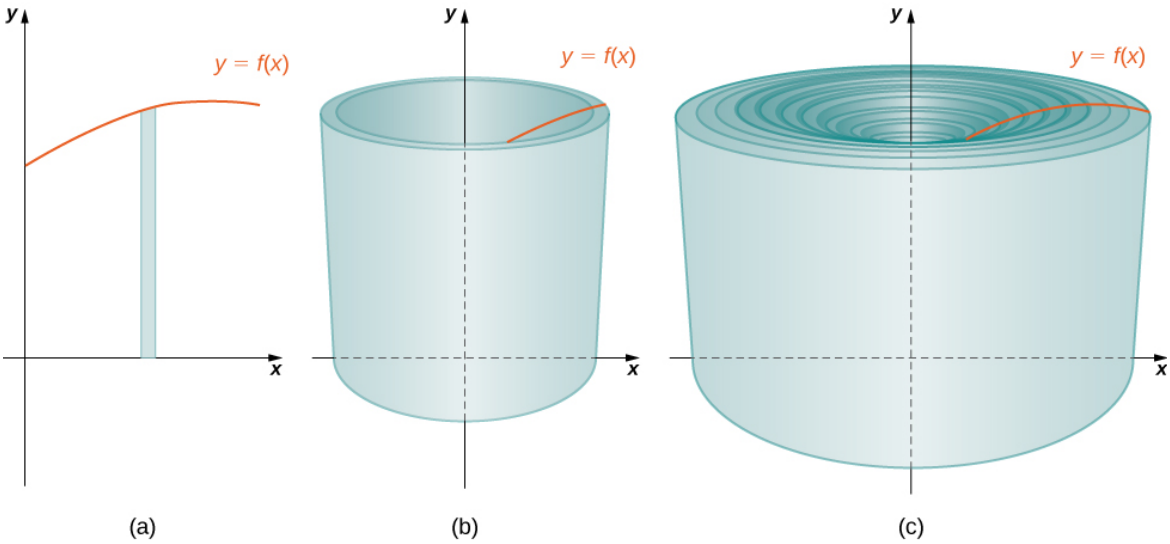


Figure 6.26 (a) A representative rectangle. (b) When this rectangle is revolved around the y-axis, the result is a cylindrical shell. (c) When we put all the shells together, we get an approximation of the original solid.

To calculate the volume of this shell, consider **Figure 6.27**.

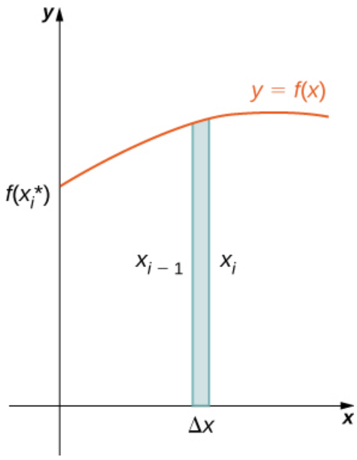


Figure 6.27 Calculating the volume of the shell.

The shell is a cylinder, so its volume is the cross-sectional area multiplied by the height of the cylinder. The cross-sections are annuli (ring-shaped regions—essentially, circles with a hole in the center), with outer radius x_i and inner radius x_{i-1}. Thus, the cross-sectional area is $\pi x_i^2 - \pi x_{i-1}^2$. The height of the cylinder is $f(x_i^*)$. Then the volume of the shell is

$$\begin{aligned} V_{\text{shell}} &= f(x_i^*)(\pi x_i^2 - \pi x_{i-1}^2) \\ &= \pi f(x_i^*)\left(x_i^2 - x_{i-1}^2\right) \\ &= \pi f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

Note that $x_i - x_{i-1} = \Delta x$, so we have

$$V_{\text{shell}} = 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x.$$

Furthermore, $\frac{x_i + x_{i-1}}{2}$ is both the midpoint of the interval $[x_{i-1}, x_i]$ and the average radius of the shell, and we can approximate this by x_i^*. We then have

$$V_{\text{shell}} \approx 2\pi f(x_i^*) x_i^* \Delta x.$$

Another way to think of this is to think of making a vertical cut in the shell and then opening it up to form a flat plate (**Figure 6.28**).

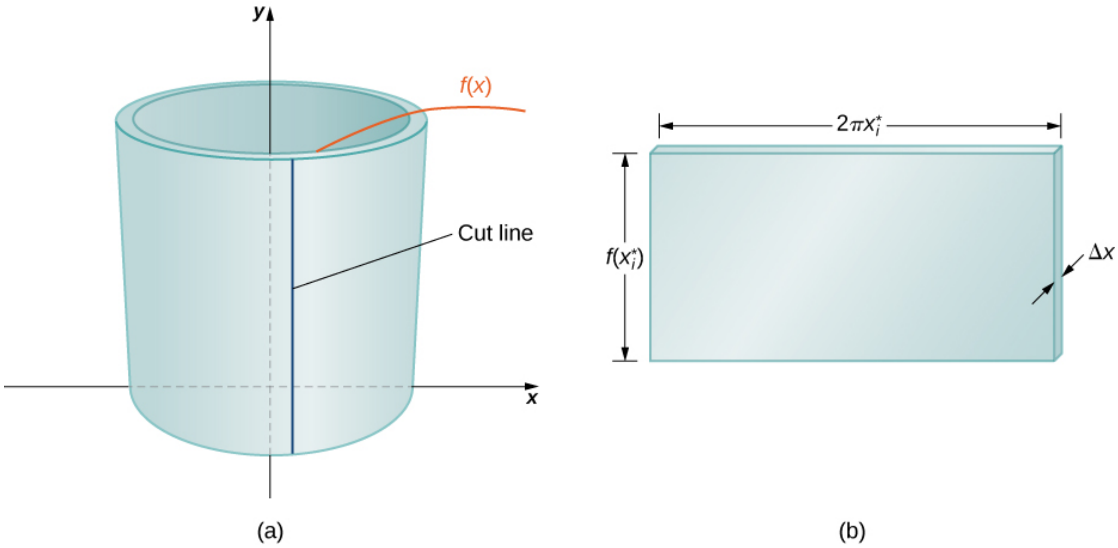


Figure 6.28 (a) Make a vertical cut in a representative shell. (b) Open the shell up to form a flat plate.

In reality, the outer radius of the shell is greater than the inner radius, and hence the back edge of the plate would be slightly longer than the front edge of the plate. However, we can approximate the flattened shell by a flat plate of height $f(x_i^*)$, width $2\pi x_i^*$, and thickness Δx (**Figure 6.28**). The volume of the shell, then, is approximately the volume of the flat plate. Multiplying the height, width, and depth of the plate, we get

$$V_{\text{shell}} \approx f(x_i^*)(2\pi x_i^*)\Delta x,$$

which is the same formula we had before.

To calculate the volume of the entire solid, we then add the volumes of all the shells and obtain

$$V \approx \sum_{i=1}^{n} \left(2\pi x_i^* f(x_i^*)\Delta x\right).$$

Here we have another Riemann sum, this time for the function $2\pi x f(x)$. Taking the limit as $n \to \infty$ gives us

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} \left(2\pi x_i^* f(x_i^*)\Delta x\right) = \int_a^b \left(2\pi x f(x)\right) dx.$$

This leads to the following rule for the **method of cylindrical shells.**

Rule: The Method of Cylindrical Shells

Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of $f(x)$, below by the x-axis, on the left by the line $x = a$, and on the right by the line $x = b$. Then the volume of the solid of revolution formed by revolving R around the y-axis is given by

$$V = \int_a^b (2\pi x f(x))dx. \tag{6.6}$$

Now let's consider an example.

Example 6.12

The Method of Cylindrical Shells 1

Define R as the region bounded above by the graph of $f(x) = 1/x$ and below by the x-axis over the interval $[1, 3]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Solution

First we must graph the region R and the associated solid of revolution, as shown in the following figure.

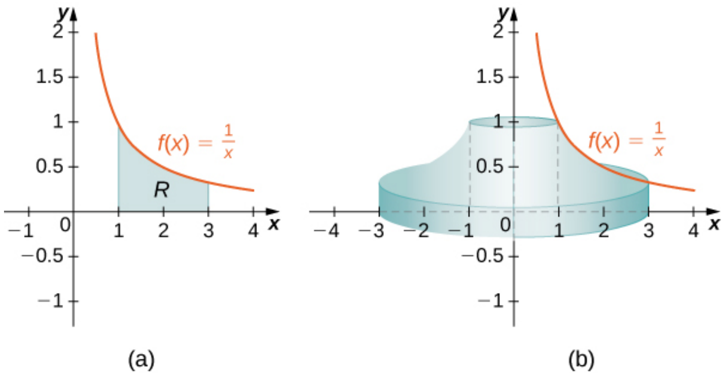


Figure 6.29 (a) The region R under the graph of $f(x) = 1/x$ over the interval $[1, 3]$. (b) The solid of revolution generated by revolving R about the y-axis.

Then the volume of the solid is given by

$$\begin{aligned} V &= \int_a^b (2\pi x f(x))dx \\ &= \int_1^3 \left(2\pi x\left(\frac{1}{x}\right)\right)dx \\ &= \int_1^3 2\pi\, dx = 2\pi x\big|_1^3 = 4\pi \text{ units}^3. \end{aligned}$$

6.12 Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Example 6.13

The Method of Cylindrical Shells 2

Define R as the region bounded above by the graph of $f(x) = 2x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Solution

First graph the region R and the associated solid of revolution, as shown in the following figure.

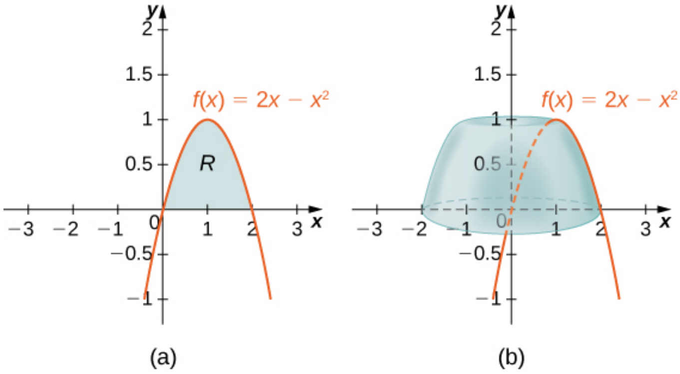


Figure 6.30 (a) The region R under the graph of $f(x) = 2x - x^2$ over the interval $[0, 2]$. (b) The volume of revolution obtained by revolving R about the y-axis.

Then the volume of the solid is given by

$$\begin{aligned} V &= \int_a^b (2\pi x f(x))dx \\ &= \int_0^2 \left(2\pi x\left(2x - x^2\right)\right)dx = 2\pi \int_0^2 \left(2x^2 - x^3\right)dx \\ &= 2\pi\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]\Bigg|_0^2 = \frac{8\pi}{3}\,\text{units}^3. \end{aligned}$$

6.13 Define R as the region bounded above by the graph of $f(x) = 3x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

As with the disk method and the washer method, we can use the method of cylindrical shells with solids of revolution, revolved around the x-axis, when we want to integrate with respect to y. The analogous rule for this type of solid is given here.

Rule: The Method of Cylindrical Shells for Solids of Revolution around the *x*-axis

Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y-axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the x-axis is given by

$$V = \int_c^d (2\pi y g(y))dy.$$

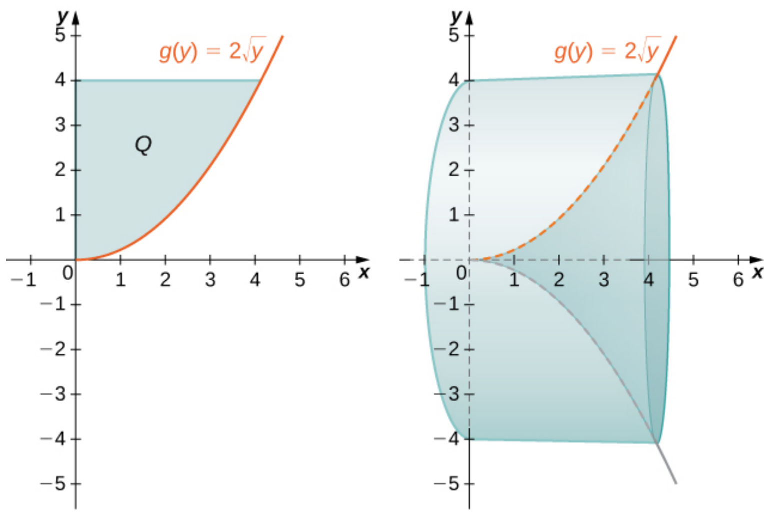
Example 6.14

The Method of Cylindrical Shells for a Solid Revolved around the *x*-axis

Define Q as the region bounded on the right by the graph of $g(y) = 2\sqrt{y}$ and on the left by the y-axis for $y \in [0, 4]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

Solution

First, we need to graph the region Q and the associated solid of revolution, as shown in the following figure.



(a) (b)

Figure 6.31 (a) The region Q to the left of the function $g(y)$ over the interval $[0, 4]$. (b) The solid of revolution generated by revolving Q around the x-axis.

Label the shaded region Q. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_c^d (2\pi y g(y))dy \\ &= \int_0^4 (2\pi y(2\sqrt{y}))dy = 4\pi \int_0^4 y^{3/2} dy \\ &= 4\pi\left[\frac{2y^{5/2}}{5}\right]\Big|_0^4 = \frac{256\pi}{5} \text{ units}^3. \end{aligned}$$

6.14 Define Q as the region bounded on the right by the graph of $g(y) = 3/y$ and on the left by the y-axis for $y \in [1, 3]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

For the next example, we look at a solid of revolution for which the graph of a function is revolved around a line other than one of the two coordinate axes. To set this up, we need to revisit the development of the method of cylindrical shells. Recall that we found the volume of one of the shells to be given by

$$\begin{aligned} V_{\text{shell}} &= f(x_i^*)(\pi x_i^2 - \pi x_{i-1}^2) \\ &= \pi f(x_i^*)\left(x_i^2 - x_{i-1}^2\right) \\ &= \pi f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

This was based on a shell with an outer radius of x_i and an inner radius of x_{i-1}. If, however, we rotate the region around a line other than the y-axis, we have a different outer and inner radius. Suppose, for example, that we rotate the region around the line $x = -k$, where k is some positive constant. Then, the outer radius of the shell is $x_i + k$ and the inner radius of the shell is $x_{i-1} + k$. Substituting these terms into the expression for volume, we see that when a plane region is rotated around the line $x = -k$, the volume of a shell is given by

$$\begin{aligned} V_{\text{shell}} &= 2\pi f(x_i^*)\left(\frac{(x_i + k) + (x_{i-1} + k)}{2}\right)\left((x_i + k) - (x_{i-1} + k)\right) \\ &= 2\pi f(x_i^*)\left(\left(\frac{x_i + x_{i-2}}{2}\right) + k\right)\Delta x. \end{aligned}$$

As before, we notice that $\dfrac{x_i + x_{i-1}}{2}$ is the midpoint of the interval $[x_{i-1}, x_i]$ and can be approximated by x_i^*. Then, the approximate volume of the shell is

$$V_{\text{shell}} \approx 2\pi\left(x_i^* + k\right)f(x_i^*)\Delta x.$$

The remainder of the development proceeds as before, and we see that

$$V = \int_a^b (2\pi(x + k)f(x))dx.$$

We could also rotate the region around other horizontal or vertical lines, such as a vertical line in the right half plane. In each case, the volume formula must be adjusted accordingly. Specifically, the x-term in the integral must be replaced with an expression representing the radius of a shell. To see how this works, consider the following example.

Example 6.15

A Region of Revolution Revolved around a Line

Define R as the region bounded above by the graph of $f(x) = x$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -1$.

Solution

First, graph the region R and the associated solid of revolution, as shown in the following figure.

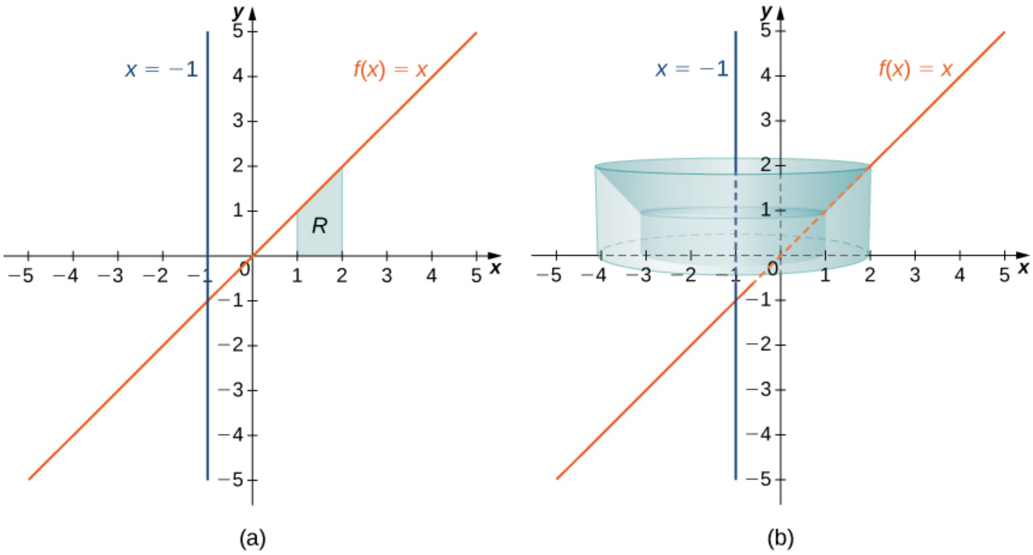


Figure 6.32 (a) The region R between the graph of $f(x)$ and the x-axis over the interval $[1, 2]$. (b) The solid of revolution generated by revolving R around the line $x = -1$.

Note that the radius of a shell is given by $x + 1$. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_1^2 (2\pi(x+1) f(x)) dx \\ &= \int_1^2 (2\pi(x+1)x) dx = 2\pi \int_1^2 \left(x^2 + x\right) dx \\ &= 2\pi \left[\frac{x^3}{3} + \frac{x^2}{2}\right]\Big|_1^2 = \frac{23\pi}{3} \text{ units}^3. \end{aligned}$$

6.15 Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -2$.

For our final example in this section, let's look at the volume of a solid of revolution for which the region of revolution is bounded by the graphs of two functions.

Example 6.16

A Region of Revolution Bounded by the Graphs of Two Functions

Define R as the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1/x$ over the interval $[1, 4]$. Find the volume of the solid of revolution generated by revolving R around the y-axis.

Solution

First, graph the region R and the associated solid of revolution, as shown in the following figure.

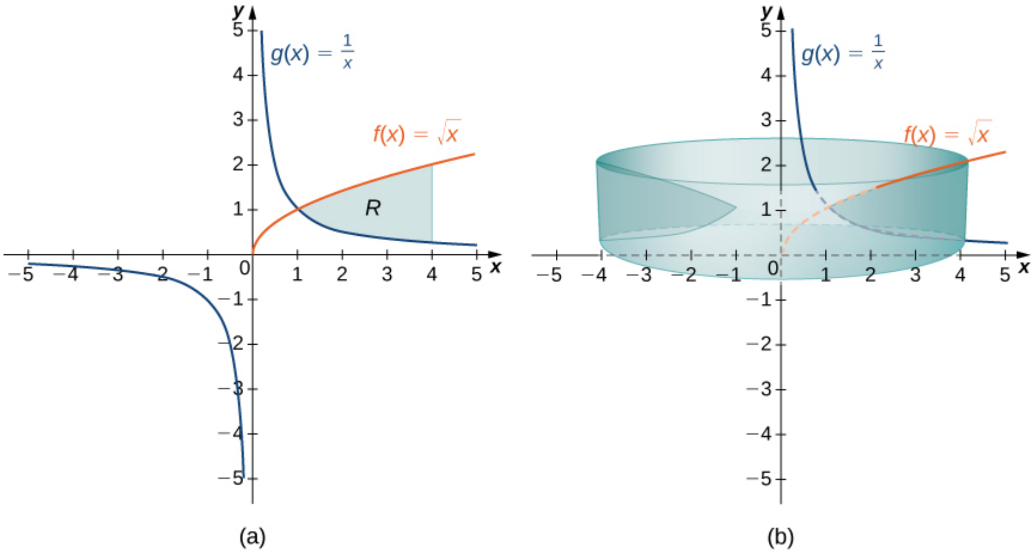


Figure 6.33 (a) The region R between the graph of $f(x)$ and the graph of $g(x)$ over the interval $[1, 4]$. (b) The solid of revolution generated by revolving R around the y-axis.

Note that the axis of revolution is the y-axis, so the radius of a shell is given simply by x. We don't need to make any adjustments to the x-term of our integrand. The height of a shell, though, is given by $f(x) - g(x)$, so in this case we need to adjust the $f(x)$ term of the integrand. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_1^4 (2\pi x(f(x) - g(x)))dx \\ &= \int_1^4 \left(2\pi x\left(\sqrt{x} - \frac{1}{x}\right)\right)dx = 2\pi \int_1^4 \left(x^{3/2} - 1\right)dx \\ &= 2\pi\left[\frac{2x^{5/2}}{5} - x\right]\Bigg|_1^4 = \frac{94\pi}{5} \text{ units}^3. \end{aligned}$$

6.16 Define R as the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = x^2$ over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Which Method Should We Use?

We have studied several methods for finding the volume of a solid of revolution, but how do we know which method to use? It often comes down to a choice of which integral is easiest to evaluate. **Figure 6.34** describes the different approaches for solids of revolution around the x-axis. It's up to you to develop the analogous table for solids of revolution around the y-axis.

Comparing the Methods for Finding the Volume of a Solid Revolution around the *x*-axis

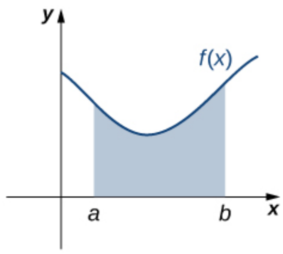
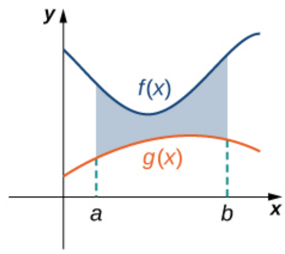
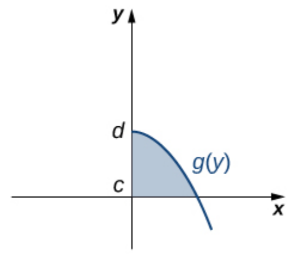
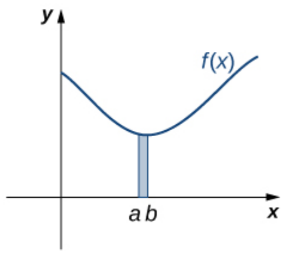
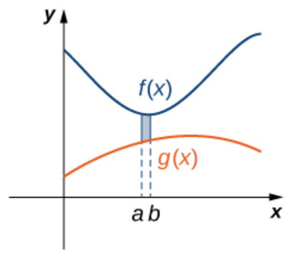
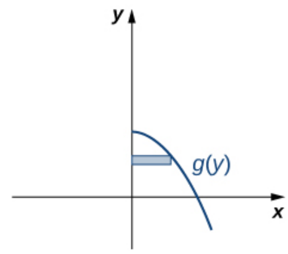
<table>
<tr><th>Compare</th><th>Disk Method</th><th>Washer Method</th><th>Shell Method</th></tr>
<tr><td>Volume formula</td><td>$V = \int_a^b \pi[f(x)]^2\,dx$</td><td>$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx$</td><td>$V = \int_c^d 2\pi y\,g(y)\,dy$</td></tr>
<tr><td>Solid</td><td>No cavity in the center</td><td>Cavity in the center</td><td>With or without a cavity in the center</td></tr>
<tr><td>Interval to partition</td><td>$[a, b]$ on x-axis</td><td>$[a, b]$ on x-axis</td><td>$[c, d]$ on y-axis</td></tr>
<tr><td>Rectangle</td><td>Vertical</td><td>Vertical</td><td>Horizontal</td></tr>
<tr><td>Typical region</td><td></td><td></td><td></td></tr>
<tr><td>Typical element</td><td></td><td></td><td></td></tr>
</table>

Figure 6.34

Let’s take a look at a couple of additional problems and decide on the best approach to take for solving them.

Example 6.17

Selecting the Best Method

For each of the following problems, select the best method to find the volume of a solid of revolution generated by revolving the given region around the x-axis, and set up the integral to find the volume (do not evaluate the integral).

a. The region bounded by the graphs of $y = x,$ $y = 2 - x,$ and the x-axis.

b. The region bounded by the graphs of $y = 4x - x^2$ and the x-axis.

Solution

a. First, sketch the region and the solid of revolution as shown.

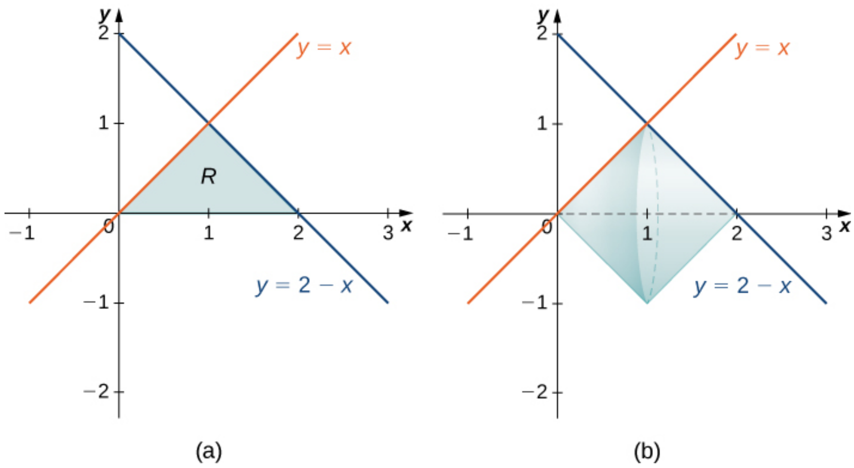


Figure 6.35 (a) The region R bounded by two lines and the x-axis. (b) The solid of revolution generated by revolving R about the x-axis.

Looking at the region, if we want to integrate with respect to x, we would have to break the integral into two pieces, because we have different functions bounding the region over $[0, 1]$ and $[1, 2]$. In this case, using the disk method, we would have

$$V = \int_0^1 \left(\pi x^2\right) dx + \int_1^2 \left(\pi (2 - x)^2\right) dx.$$

If we used the shell method instead, we would use functions of y to represent the curves, producing

$$\begin{aligned} V &= \int_0^1 (2\pi y[(2 - y) - y]) dy \\ &= \int_0^1 (2\pi y[2 - 2y]) dy. \end{aligned}$$

Neither of these integrals is particularly onerous, but since the shell method requires only one integral, and the integrand requires less simplification, we should probably go with the shell method in this case.

b. First, sketch the region and the solid of revolution as shown.

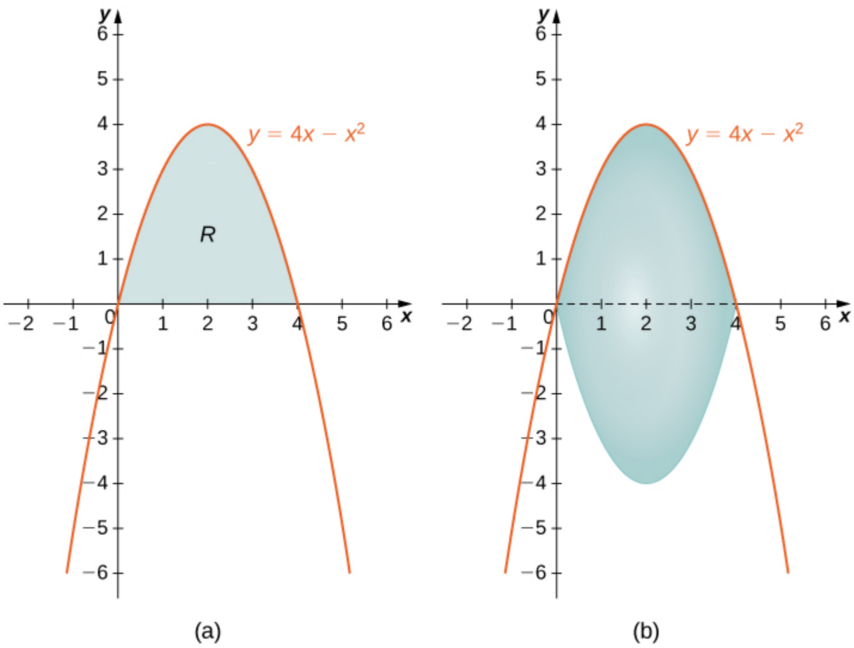


Figure 6.36 (a) The region R between the curve and the x-axis. (b) The solid of revolution generated by revolving R about the x-axis.

Looking at the region, it would be problematic to define a horizontal rectangle; the region is bounded on the left and right by the same function. Therefore, we can dismiss the method of shells. The solid has no cavity in the middle, so we can use the method of disks. Then

$$V = \int_0^4 \pi \left(4x - x^2\right)^2 dx.$$

6.17 Select the best method to find the volume of a solid of revolution generated by revolving the given region around the x-axis, and set up the integral to find the volume (do not evaluate the integral): the region bounded by the graphs of $y = 2 - x^2$ and $y = x^2$.

6.3 EXERCISES

For the following exercise, find the volume generated when the region between the two curves is rotated around the given axis. Use both the shell method and the washer method. Use technology to graph the functions and draw a typical slice by hand.

114. **[T]** Over the curve of $y = 3x$, $x = 0$, and $y = 3$ rotated around the y-axis.

115. **[T]** Under the curve of $y = 3x$, $y = 0$, and $x = 3$ rotated around the y-axis.

116. **[T]** Over the curve of $y = 3x$, $y = 0$, and $y = 3$ rotated around the x-axis.

117. **[T]** Under the curve of $y = 3x$, $y = 0$, and $x = 3$ rotated around the x-axis.

118. **[T]** Under the curve of $y = 2x^3$, $x = 0$, and $x = 2$ rotated around the y-axis.

119. **[T]** Under the curve of $y = 2x^3$, $x = 0$, and $x = 2$ rotated around the x-axis.

For the following exercises, use shells to find the volumes of the given solids. Note that the rotated regions lie between the curve and the x-axis and are rotated around the y-axis.

120. $y = 1 - x^2$, $x = 0$, and $x = 1$

121. $y = 5x^3$, $x = 0$, and $x = 1$

122. $y = \frac{1}{x}$, $x = 1$, and $x = 100$

123. $y = \sqrt{1 - x^2}$, $x = 0$, and $x = 1$

124. $y = \frac{1}{1 + x^2}$, $x = 0$, and $x = 3$

125. $y = \sin x^2$, $x = 0$, and $x = \sqrt{\pi}$

126. $y = \frac{1}{\sqrt{1 - x^2}}$, $x = 0$, and $x = \frac{1}{2}$

127. $y = \sqrt{x}$, $x = 0$, and $x = 1$

128. $y = \left(1 + x^2\right)^3$, $x = 0$, and $x = 1$

129. $y = 5x^3 - 2x^4$, $x = 0$, and $x = 2$

For the following exercises, use shells to find the volume generated by rotating the regions between the given curve and $y = 0$ around the x-axis.

130. $y = \sqrt{1 - x^2}$, $x = 0$, $x = 1$ and the x-axis

131. $y = x^2$, $x = 0$, $x = 2$ and the x-axis

132. $y = \frac{x^3}{2}$, $x = 0$, $x = 2$, and the x-axis

133. $y = \frac{2}{x^2}$, $x = 1$, $x = 2$, and the x-axis

134. $x = \frac{1}{1 + y^2}$, $x = \frac{1}{5}$, and $y = 0$

135. $x = \frac{1 + y^2}{y}$, $y = 1$, $y = 4$, and the y-axis

136. $x = \cos y$, $y = 0$, and $y = \pi$

137. $x = y^3 - 2y^2$, $x = 0$, $x = 9$, and the y-axis

138. $x = \sqrt{y} + 1$, $x = 1$, $x = 3$, and the x-axis

139. $x = \sqrt[3]{27y}$ and $x = \frac{3y}{4}$

For the following exercises, find the volume generated when the region between the curves is rotated around the given axis.

140. $y = 3 - x$, $y = 0$, $x = 0$, and $x = 2$ rotated around the y-axis.

141. $y = x^3$, $x = 0$, and $y = 8$ rotated around the y-axis.

142. $y = x^2$, $y = x$, rotated around the y-axis.

143. $y = \sqrt{x}$, $y = 0$, and $x = 1$ rotated around the line $x = 2$.

144. $y = \frac{1}{4 - x}$, $x = 1$, and $x = 2$ rotated around the line $x = 4$.

145. $y = \sqrt{x}$ and $y = x^2$ rotated around the y-axis.

146. $y = \sqrt{x}$ and $y = x^2$ rotated around the line $x = 2$.

147. $x = y^3$, $x = \frac{1}{y}$, $x = 1$, and $x = 2$ rotated around the x-axis.

148. $x = y^2$ and $y = x$ rotated around the line $y = 2$.

149. **[T]** Left of $x = \sin(\pi y)$, right of $y = x$, around the y-axis.

For the following exercises, use technology to graph the region. Determine which method you think would be easiest to use to calculate the volume generated when the function is rotated around the specified axis. Then, use your chosen method to find the volume.

150. **[T]** $y = x^2$ and $y = 4x$ rotated around the y-axis.

151. **[T]** $y = \cos(\pi x)$, $y = \sin(\pi x)$, $x = \frac{1}{4}$, and $x = \frac{5}{4}$ rotated around the y-axis.

152. **[T]** $y = x^2 - 2x$, $x = 2$, and $x = 4$ rotated around the y-axis.

153. **[T]** $y = x^2 - 2x$, $x = 2$, and $x = 4$ rotated around the x-axis.

154. **[T]** $y = 3x^3 - 2$, $y = x$, and $x = 2$ rotated around the x-axis.

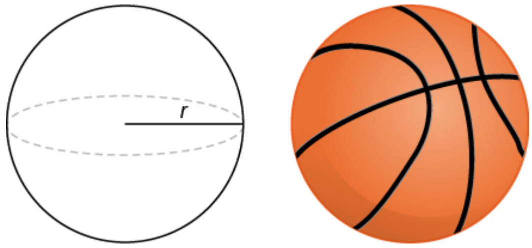
155. **[T]** $y = 3x^3 - 2$, $y = x$, and $x = 2$ rotated around the y-axis.

156. **[T]** $x = \sin\left(\pi y^2\right)$ and $x = \sqrt{2}y$ rotated around the x-axis.

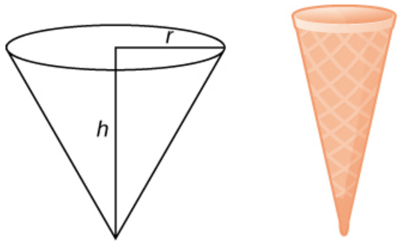
157. **[T]** $x = y^2$, $x = y^2 - 2y + 1$, and $x = 2$ rotated around the y-axis.

For the following exercises, use the method of shells to approximate the volumes of some common objects, which are pictured in accompanying figures.

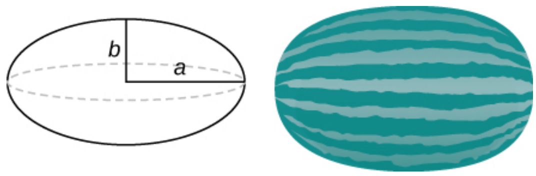
158. Use the method of shells to find the volume of a sphere of radius r.



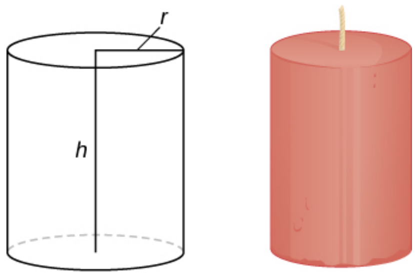
159. Use the method of shells to find the volume of a cone with radius r and height h.



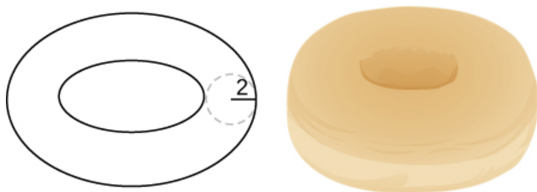
160. Use the method of shells to find the volume of an ellipse $\left(x^2/a^2\right) + \left(y^2/b^2\right) = 1$ rotated around the x-axis.



161. Use the method of shells to find the volume of a cylinder with radius r and height h.



162. Use the method of shells to find the volume of the donut created when the circle $x^2 + y^2 = 4$ is rotated around the line $x = 4$.



163. Consider the region enclosed by the graphs of $y = f(x),\ y = 1 + f(x),\ x = 0,\ y = 0,$ and $x = a > 0.$ What is the volume of the solid generated when this region is rotated around the y-axis? Assume that the function is defined over the interval $[0, a].$

164. Consider the function $y = f(x),$ which decreases from $f(0) = b$ to $f(1) = 0.$ Set up the integrals for determining the volume, using both the shell method and the disk method, of the solid generated when this region, with $x = 0$ and $y = 0,$ is rotated around the y-axis. Prove that both methods approximate the same volume. Which method is easier to apply? (*Hint:* Since $f(x)$ is one-to-one, there exists an inverse $f^{-1}(y)$.)

6.4 | Arc Length of a Curve and Surface Area

Learning Objectives

6.4.1 Determine the length of a curve, $y = f(x)$, between two points.

6.4.2 Determine the length of a curve, $x = g(y)$, between two points.

6.4.3 Find the surface area of a solid of revolution.

In this section, we use definite integrals to find the arc length of a curve. We can think of **arc length** as the distance you would travel if you were walking along the path of the curve. Many real-world applications involve arc length. If a rocket is launched along a parabolic path, we might want to know how far the rocket travels. Or, if a curve on a map represents a road, we might want to know how far we have to drive to reach our destination.

We begin by calculating the arc length of curves defined as functions of x, then we examine the same process for curves defined as functions of y. (The process is identical, with the roles of x and y reversed.) The techniques we use to find arc length can be extended to find the surface area of a surface of revolution, and we close the section with an examination of this concept.

Arc Length of the Curve y = f(x)

In previous applications of integration, we required the function $f(x)$ to be integrable, or at most continuous. However, for calculating arc length we have a more stringent requirement for $f(x)$. Here, we require $f(x)$ to be differentiable, and furthermore we require its derivative, $f'(x)$, to be continuous. Functions like this, which have continuous derivatives, are called *smooth*. (This property comes up again in later chapters.)

Let $f(x)$ be a smooth function defined over $[a, b]$. We want to calculate the length of the curve from the point $(a, f(a))$ to the point $(b, f(b))$. We start by using line segments to approximate the length of the curve. For $i = 0, 1, 2,\ldots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2,\ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Although it might seem logical to use either horizontal or vertical line segments, we want our line segments to approximate the curve as closely as possible. Figure 6.37 depicts this construct for $n = 5$.

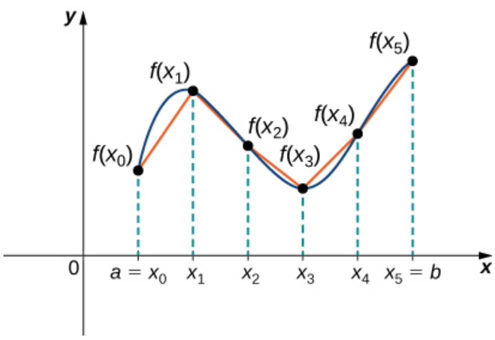


Figure 6.37 We can approximate the length of a curve by adding line segments.

To help us find the length of each line segment, we look at the change in vertical distance as well as the change in horizontal distance over each interval. Because we have used a regular partition, the change in horizontal distance over each interval is given by Δx. The change in vertical distance varies from interval to interval, though, so we use $\Delta y_i = f(x_i) - f(x_{i-1})$ to represent the change in vertical distance over the interval $[x_{i-1}, x_i]$, as shown in Figure 6.38. Note that some (or all) Δy_i may be negative.

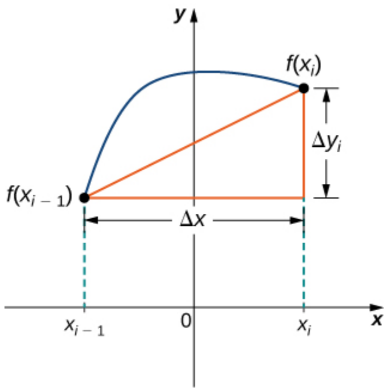


Figure 6.38 A representative line segment approximates the curve over the interval $[x_{i-1}, x_i]$.

By the Pythagorean theorem, the length of the line segment is $\sqrt{(\Delta x)^2 + (\Delta y_i)^2}$. We can also write this as $\Delta x\sqrt{1 + ((\Delta y_i)/(\Delta x))^2}$. Now, by the Mean Value Theorem, there is a point $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = (\Delta y_i)/(\Delta x)$. Then the length of the line segment is given by $\Delta x\sqrt{1 + [f'(x_i^*)]^2}$. Adding up the lengths of all the line segments, we get

$$\text{Arc Length} \approx \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we have

$$\text{Arc Length} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

We summarize these findings in the following theorem.

Theorem 6.4: Arc Length for *y* = *f*(*x*)

Let $f(x)$ be a smooth function over the interval $[a, b]$. Then the arc length of the portion of the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx. \tag{6.7}$$

Note that we are integrating an expression involving $f'(x)$, so we need to be sure $f'(x)$ is integrable. This is why we require $f(x)$ to be smooth. The following example shows how to apply the theorem.

Example 6.18

Calculating the Arc Length of a Function of *x*

Let $f(x) = 2x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.

Solution

We have $f'(x) = 3x^{1/2}$, so $[f'(x)]^2 = 9x$. Then, the arc length is

$$\begin{aligned}\text{Arc Length} &= \int_a^b \sqrt{1 + [f'(x)]^2}\,dx \\ &= \int_0^1 \sqrt{1 + 9x}\,dx.\end{aligned}$$

Substitute $u = 1 + 9x$. Then, $du = 9\,dx$. When $x = 0$, then $u = 1$, and when $x = 1$, then $u = 10$. Thus,

$$\begin{aligned}\text{Arc Length} &= \int_0^1 \sqrt{1 + 9x}\,dx \\ &= \frac{1}{9}\int_0^1 \sqrt{1 + 9x}9dx = \frac{1}{9}\int_1^{10} \sqrt{u}\,du \\ &= \frac{1}{9}\cdot\frac{2}{3}u^{3/2}\Big|_1^{10} = \frac{2}{27}\left[10\sqrt{10} - 1\right] \approx 2.268 \text{ units.}\end{aligned}$$

6.18 Let $f(x) = (4/3)x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.

Although it is nice to have a formula for calculating arc length, this particular theorem can generate expressions that are difficult to integrate. We study some techniques for integration in **Introduction to Techniques of Integration (http://cnx.org/content/m53654/latest/)** . In some cases, we may have to use a computer or calculator to approximate the value of the integral.

Example 6.19

Using a Computer or Calculator to Determine the Arc Length of a Function of *x*

Let $f(x) = x^2$. Calculate the arc length of the graph of $f(x)$ over the interval $[1, 3]$.

Solution

We have $f'(x) = 2x$, so $[f'(x)]^2 = 4x^2$. Then the arc length is given by

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx = \int_1^3 \sqrt{1 + 4x^2}\,dx.$$

Using a computer to approximate the value of this integral, we get

$$\int_1^3 \sqrt{1 + 4x^2}\,dx \approx 8.26815.$$

6.19 Let $f(x) = \sin x$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, \pi]$. Use a computer or calculator to approximate the value of the integral.

Arc Length of the Curve $x = g(y)$

We have just seen how to approximate the length of a curve with line segments. If we want to find the arc length of the graph of a function of $y,$ we can repeat the same process, except we partition the y-axis instead of the x-axis. Figure 6.39 shows a representative line segment.

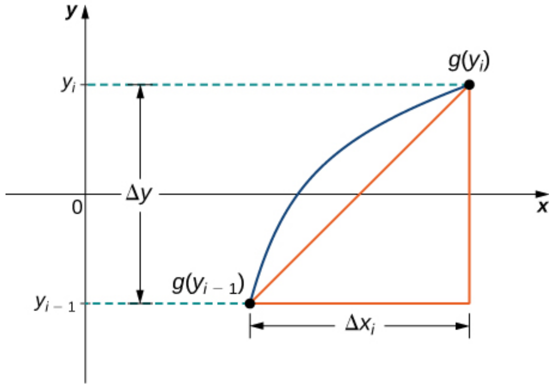


Figure 6.39 A representative line segment over the interval $[y_{i-1}, y_i]$.

Then the length of the line segment is $\sqrt{(\Delta y)^2 + (\Delta x_i)^2},$ which can also be written as $\Delta y \sqrt{1 + ((\Delta x_i)/(\Delta y))^2}.$ If we now follow the same development we did earlier, we get a formula for arc length of a function $x = g(y).$

Theorem 6.5: Arc Length for $x = g(y)$

Let $g(y)$ be a smooth function over an interval $[c, d].$ Then, the arc length of the graph of $g(y)$ from the point $(c, g(c))$ to the point $(d, g(d))$ is given by

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \quad (6.8)$$

Example 6.20

Calculating the Arc Length of a Function of y

Let $g(y) = 3y^3.$ Calculate the arc length of the graph of $g(y)$ over the interval $[1, 2].$

Solution

We have $g'(y) = 9y^2,$ so $[g'(y)]^2 = 81y^4.$ Then the arc length is

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_1^2 \sqrt{1 + 81y^4}\, dy.$$

Using a computer to approximate the value of this integral, we obtain

$$\int_1^2 \sqrt{1 + 81y^4}\, dy \approx 21.0277.$$

6.20 Let $g(y) = 1/y$. Calculate the arc length of the graph of $g(y)$ over the interval $[1, 4]$. Use a computer or calculator to approximate the value of the integral.

Area of a Surface of Revolution

The concepts we used to find the arc length of a curve can be extended to find the surface area of a surface of revolution. **Surface area** is the total area of the outer layer of an object. For objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces. For curved surfaces, the situation is a little more complex. Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the x-axis as shown in the following figure.

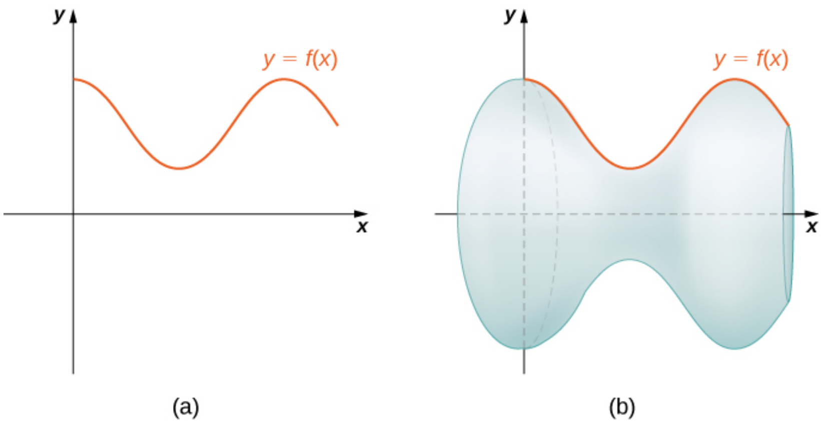


Figure 6.40 (a) A curve representing the function $f(x)$. (b) The surface of revolution formed by revolving the graph of $f(x)$ around the x-axis.

As we have done many times before, we are going to partition the interval $[a, b]$ and approximate the surface area by calculating the surface area of simpler shapes. We start by using line segments to approximate the curve, as we did earlier in this section. For $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of $[a, b]$. Then, for $i = 1, 2,\ldots, n,$ construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Now, revolve these line segments around the x-axis to generate an approximation of the surface of revolution as shown in the following figure.

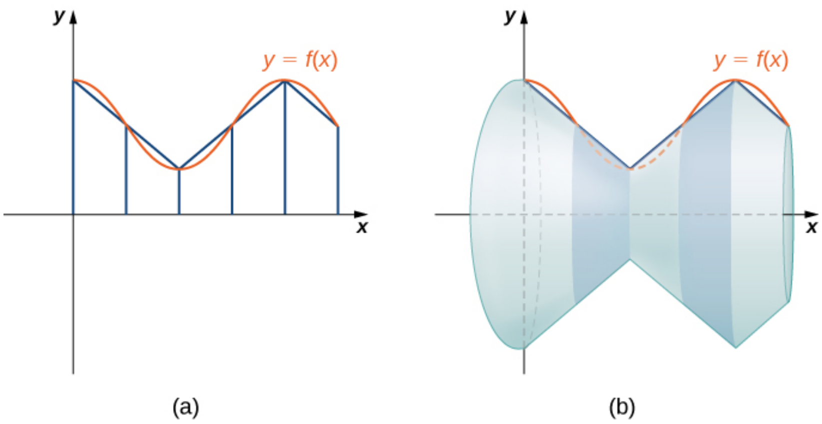


Figure 6.41 (a) Approximating $f(x)$ with line segments. (b) The surface of revolution formed by revolving the line segments around the x-axis.

Notice that when each line segment is revolved around the axis, it produces a band. These bands are actually pieces of cones (think of an ice cream cone with the pointy end cut off). A piece of a cone like this is called a **frustum** of a cone.

To find the surface area of the band, we need to find the lateral surface area, S, of the frustum (the area of just the slanted outside surface of the frustum, not including the areas of the top or bottom faces). Let r_1 and r_2 be the radii of the wide end and the narrow end of the frustum, respectively, and let l be the slant height of the frustum as shown in the following figure.

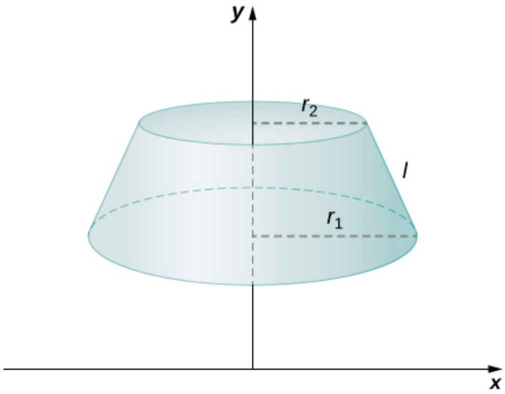


Figure 6.42 A frustum of a cone can approximate a small part of surface area.

We know the lateral surface area of a cone is given by

$$\text{Lateral Surface Area} = \pi rs,$$

where r is the radius of the base of the cone and s is the slant height (see the following figure).

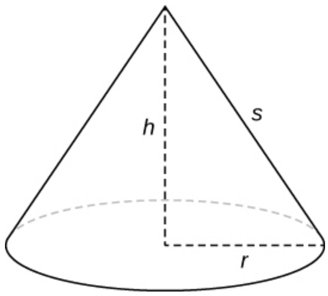


Figure 6.43 The lateral surface area of the cone is given by πrs.

Since a frustum can be thought of as a piece of a cone, the lateral surface area of the frustum is given by the lateral surface area of the whole cone less the lateral surface area of the smaller cone (the pointy tip) that was cut off (see the following figure).

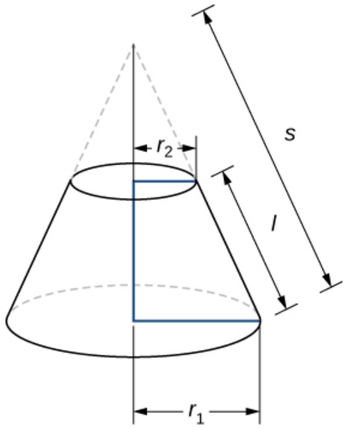


Figure 6.44 Calculating the lateral surface area of a frustum of a cone.

The cross-sections of the small cone and the large cone are similar triangles, so we see that

$$\frac{r_2}{r_1} = \frac{s-l}{s}.$$

Solving for s, we get

$$\begin{aligned}
\frac{r_2}{r_1} &= \frac{s-l}{s} \\
r_2 s &= r_1(s-l) \\
r_2 s &= r_1 s - r_1 l \\
r_1 l &= r_1 s - r_2 s \\
r_1 l &= (r_1 - r_2)s \\
\frac{r_1 l}{r_1 - r_2} &= s.
\end{aligned}$$

Then the lateral surface area (SA) of the frustum is

$$\begin{aligned}
S &= \text{(Lateral SA of large cone)} - \text{(Lateral SA of small cone)} \\
&= \pi r_1 s - \pi r_2(s-l) \\
&= \pi r_1\left(\frac{r_1 l}{r_1 - r_2}\right) - \pi r_2\left(\frac{r_1 l}{r_1 - r_2} - l\right) \\
&= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \pi r_2 l \\
&= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \frac{\pi r_2 l(r_1 - r_2)}{r_1 - r_2} \\
&= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \frac{\pi r_1 r_2 l}{r_1 - r_2} - \frac{\pi r_2{}^2 l}{r_1 - r_2} \\
&= \frac{\pi\left(r_1^2 - r_2^2\right)l}{r_1 - r_2} = \frac{\pi(r_1 - r_2)(r_1 + r_2)l}{r_1 - r_2} = \pi(r_1 + r_2)l.
\end{aligned}$$

Let's now use this formula to calculate the surface area of each of the bands formed by revolving the line segments around the x-axis. A representative band is shown in the following figure.

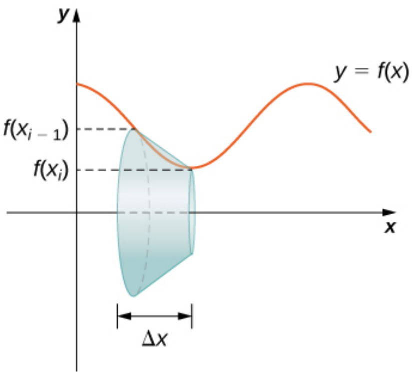


Figure 6.45 A representative band used for determining surface area.

Note that the slant height of this frustum is just the length of the line segment used to generate it. So, applying the surface area formula, we have

$$\begin{aligned} S &= \pi(r_1 + r_2)l \\ &= \pi\left(f(x_{i-1}) + f(x_i)\right)\sqrt{\Delta x^2 + (\Delta y_i)^2} \\ &= \pi\left(f(x_{i-1}) + f(x_i)\right)\Delta x\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}. \end{aligned}$$

Now, as we did in the development of the arc length formula, we apply the Mean Value Theorem to select $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = (\Delta y_i)/\Delta x$. This gives us

$$S = \pi\left(f(x_{i-1}) + f(x_i)\right)\Delta x\sqrt{1 + \left(f'(x_i^*)\right)^2}.$$

Furthermore, since $f(x)$ is continuous, by the Intermediate Value Theorem, there is a point $x_i^{**} \in [x_{i-1}, x_i]$ such that $f(x_i^{**}) = (1/2)\left[f(x_{i-1}) + f(x_i)\right]$, so we get

$$S = 2\pi f(x_i^{**})\Delta x\sqrt{1 + \left(f'(x_i^*)\right)^2}.$$

Then the approximate surface area of the whole surface of revolution is given by

$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1 + \left(f'(x_i^*)\right)^2}.$$

This *almost* looks like a Riemann sum, except we have functions evaluated at two different points, x_i^* and x_i^{**}, over the interval $[x_{i-1}, x_i]$. Although we do not examine the details here, it turns out that because $f(x)$ is smooth, if we let $n \to \infty$, the limit works the same as a Riemann sum even with the two different evaluation points. This makes sense intuitively. Both x_i^* and x_i^{**} are in the interval $[x_{i-1}, x_i]$, so it makes sense that as $n \to \infty$, both x_i^* and x_i^{**} approach x. Those of you who are interested in the details should consult an advanced calculus text.

Taking the limit as $n \to \infty$, we get

$$\text{Surface Area} = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1 + \left(f'(x_i^*)\right)^2} = \int_a^b \left(2\pi f(x)\sqrt{1 + \left(f'(x)\right)^2}\right)dx.$$

As with arc length, we can conduct a similar development for functions of y to get a formula for the surface area of surfaces of revolution about the y-axis. These findings are summarized in the following theorem.

Theorem 6.6: Surface Area of a Surface of Revolution

Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. Then, the surface area of the surface of revolution formed by revolving the graph of $f(x)$ around the x-axis is given by

$$\text{Surface Area} = \int_a^b \left(2\pi f(x)\sqrt{1 + (f'(x))^2}\right)dx. \tag{6.9}$$

Similarly, let $g(y)$ be a nonnegative smooth function over the interval $[c, d]$. Then, the surface area of the surface of revolution formed by revolving the graph of $g(y)$ around the y-axis is given by

$$\text{Surface Area} = \int_c^d \left(2\pi g(y)\sqrt{1 + (g'(y))^2}\right)dy.$$

Example 6.21

Calculating the Surface Area of a Surface of Revolution 1

Let $f(x) = \sqrt{x}$ over the interval $[1, 4]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Solution

The graph of $f(x)$ and the surface of rotation are shown in the following figure.

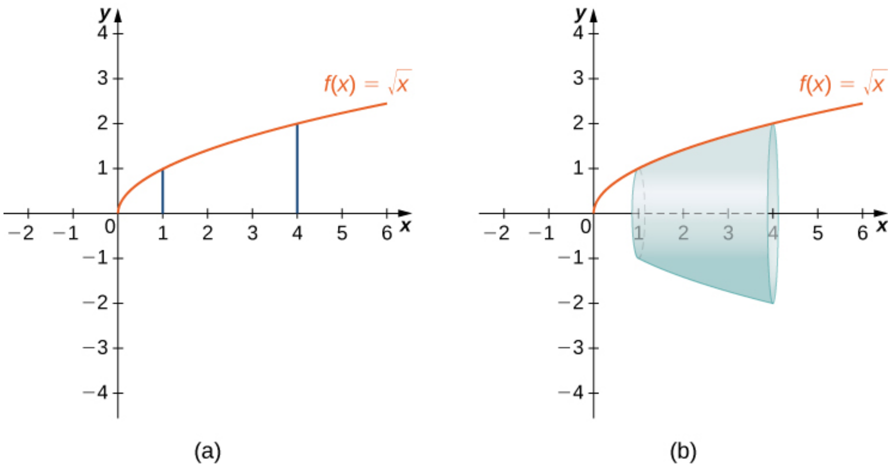


Figure 6.46 (a) The graph of $f(x)$. (b) The surface of revolution.

We have $f(x) = \sqrt{x}$. Then, $f'(x) = 1/(2\sqrt{x})$ and $(f'(x))^2 = 1/(4x)$. Then,

$$\begin{aligned}\text{Surface Area} &= \int_a^b \left(2\pi f(x)\sqrt{1+(f'(x))^2}\right)dx \\ &= \int_1^4 \left(2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\right)dx \\ &= \int_1^4 \left(2\pi\sqrt{x+\frac{1}{4}}\right)dx.\end{aligned}$$

Let $u = x + 1/4$. Then, $du = dx$. When $x = 1$, $u = 5/4$, and when $x = 4$, $u = 17/4$. This gives us

$$\begin{aligned}\int_1^4 \left(2\pi\sqrt{x+\frac{1}{4}}\right)dx &= \int_{5/4}^{17/4} 2\pi\sqrt{u}\,du \\ &= 2\pi\left[\frac{2}{3}u^{3/2}\right]\Big|_{5/4}^{17/4} = \frac{\pi}{6}\left[17\sqrt{17} - 5\sqrt{5}\right] \approx 30.846.\end{aligned}$$

6.21 Let $f(x) = \sqrt{1-x}$ over the interval $[0, 1/2]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Example 6.22

Calculating the Surface Area of a Surface of Revolution 2

Let $f(x) = y = \sqrt[3]{3x}$. Consider the portion of the curve where $0 \le y \le 2$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the y-axis.

Solution

Notice that we are revolving the curve around the y-axis, and the interval is in terms of y, so we want to rewrite the function as a function of y. We get $x = g(y) = (1/3)y^3$. The graph of $g(y)$ and the surface of rotation are shown in the following figure.

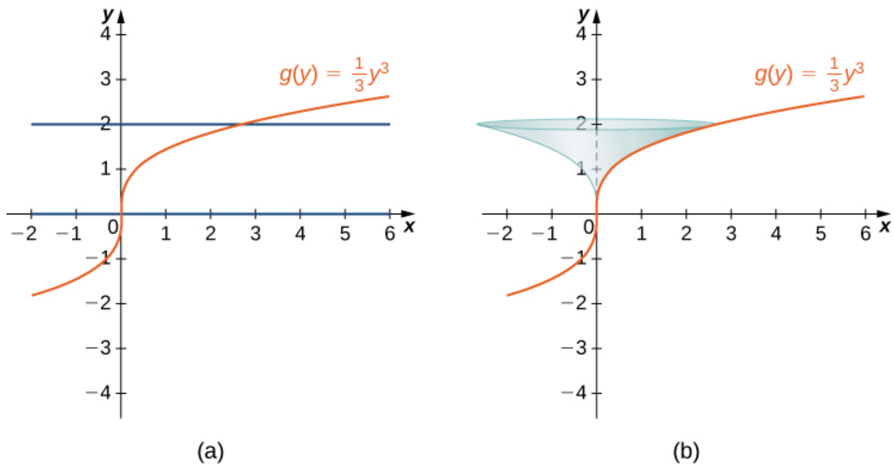


Figure 6.47 (a) The graph of $g(y)$. (b) The surface of revolution.

We have $g(y) = (1/3)y^3$, so $g'(y) = y^2$ and $(g'(y))^2 = y^4$. Then

$$\begin{aligned}\text{Surface Area} &= \int_c^d \left(2\pi g(y)\sqrt{1 + (g'(y))^2}\right)dy \\ &= \int_0^2 \left(2\pi\left(\frac{1}{3}y^3\right)\sqrt{1 + y^4}\right)dy \\ &= \frac{2\pi}{3}\int_0^2 \left(y^3\sqrt{1 + y^4}\right)dy.\end{aligned}$$

Let $u = y^4 + 1$. Then $du = 4y^3\,dy$. When $y = 0$, $u = 1$, and when $y = 2$, $u = 17$. Then

$$\begin{aligned}\frac{2\pi}{3}\int_0^2 \left(y^3\sqrt{1 + y^4}\right)dy &= \frac{2\pi}{3}\int_1^{17} \frac{1}{4}\sqrt{u}\,du \\ &= \frac{\pi}{6}\left[\frac{2}{3}u^{3/2}\right]\Big|_1^{17} = \frac{\pi}{9}\left[(17)^{3/2} - 1\right] \approx 24.118.\end{aligned}$$

6.22 Let $g(y) = \sqrt{9 - y^2}$ over the interval $y \in [0, 2]$. Find the surface area of the surface generated by revolving the graph of $g(y)$ around the y-axis.

6.4 EXERCISES

For the following exercises, find the length of the functions over the given interval.

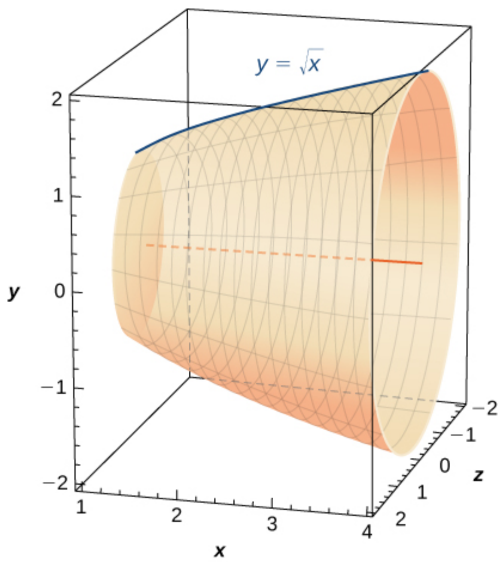
165. $y = 5x$ from $x = 0$ to $x = 2$

166. $y = -\frac{1}{2}x + 25$ from $x = 1$ to $x = 4$

167. $x = 4y$ from $y = -1$ to $y = 1$

168. Pick an arbitrary linear function $x = g(y)$ over any interval of your choice (y_1, y_2). Determine the length of the function and then prove the length is correct by using geometry.

169. Find the surface area of the volume generated when the curve $y = \sqrt{x}$ revolves around the x-axis from $(1, 1)$ to $(4, 2)$, as seen here.



170. Find the surface area of the volume generated when the curve $y = x^2$ revolves around the y-axis from $(1, 1)$ to $(3, 9)$.

For the following exercises, find the lengths of the functions of x over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

171. $y = x^{3/2}$ from $(0, 0)$ to $(1, 1)$

172. $y = x^{2/3}$ from $(1, 1)$ to $(8, 4)$

173. $y = \frac{1}{3}\left(x^2 + 2\right)^{3/2}$ from $x = 0$ to $x = 1$

174. $y = \frac{1}{3}\left(x^2 - 2\right)^{3/2}$ from $x = 2$ to $x = 4$

175. **[T]** $y = e^x$ on $x = 0$ to $x = 1$

176. $y = \frac{x^3}{3} + \frac{1}{4x}$ from $x = 1$ to $x = 3$

177. $y = \frac{x^4}{4} + \frac{1}{8x^2}$ from $x = 1$ to $x = 2$

178. $y = \frac{2x^{3/2}}{3} - \frac{x^{1/2}}{2}$ from $x = 1$ to $x = 4$

179. $y = \frac{1}{27}\left(9x^2 + 6\right)^{3/2}$ from $x = 0$ to $x = 2$

180. **[T]** $y = \sin x$ on $x = 0$ to $x = \pi$

For the following exercises, find the lengths of the functions of y over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

181. $y = \frac{5 - 3x}{4}$ from $y = 0$ to $y = 4$

182. $x = \frac{1}{2}\left(e^y + e^{-y}\right)$ from $y = -1$ to $y = 1$

183. $x = 5y^{3/2}$ from $y = 0$ to $y = 1$

184. **[T]** $x = y^2$ from $y = 0$ to $y = 1$

185. $x = \sqrt{y}$ from $y = 0$ to $y = 1$

186. $x = \frac{2}{3}\left(y^2 + 1\right)^{3/2}$ from $y = 1$ to $y = 3$

187. **[T]** $x = \tan y$ from $y = 0$ to $y = \frac{3}{4}$

188. **[T]** $x = \cos^2 y$ from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$

189. **[T]** $x = 4^y$ from $y = 0$ to $y = 2$

190. **[T]** $x = \ln(y)$ on $y = \frac{1}{e}$ to $y = e$

For the following exercises, find the surface area of the volume generated when the following curves revolve around the x-axis. If you cannot evaluate the integral exactly, use your calculator to approximate it.

191. $y = \sqrt{x}$ from $x = 2$ to $x = 6$

192. $y = x^3$ from $x = 0$ to $x = 1$

193. $y = 7x$ from $x = -1$ to $x = 1$

194. **[T]** $y = \frac{1}{x^2}$ from $x = 1$ to $x = 3$

195. $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$

196. $y = \sqrt{4 - x^2}$ from $x = -1$ to $x = 1$

197. $y = 5x$ from $x = 1$ to $x = 5$

198. **[T]** $y = \tan x$ from $x = -\frac{\pi}{4}$ to $x = \frac{\pi}{4}$

For the following exercises, find the surface area of the volume generated when the following curves revolve around the y-axis. If you cannot evaluate the integral exactly, use your calculator to approximate it.

199. $y = x^2$ from $x = 0$ to $x = 2$

200. $y = \frac{1}{2}x^2 + \frac{1}{2}$ from $x = 0$ to $x = 1$

201. $y = x + 1$ from $x = 0$ to $x = 3$

202. **[T]** $y = \frac{1}{x}$ from $x = \frac{1}{2}$ to $x = 1$

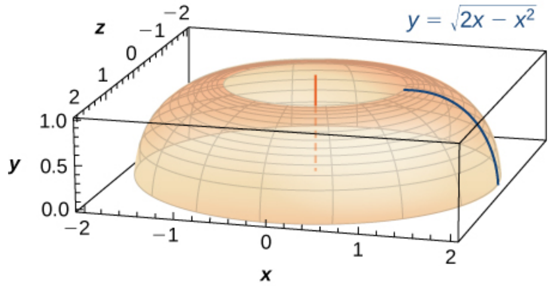
203. $y = \sqrt[3]{x}$ from $x = 1$ to $x = 27$

204. **[T]** $y = 3x^4$ from $x = 0$ to $x = 1$

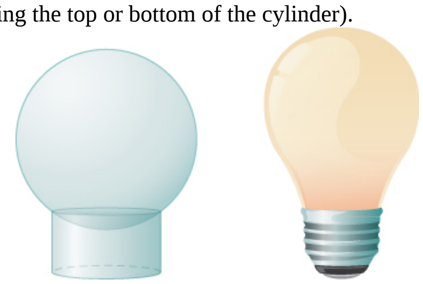
205. **[T]** $y = \frac{1}{\sqrt{x}}$ from $x = 1$ to $x = 3$

206. **[T]** $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$

207. The base of a lamp is constructed by revolving a quarter circle $y = \sqrt{2x - x^2}$ around the y-axis from $x = 1$ to $x = 2$, as seen here. Create an integral for the surface area of this curve and compute it.



208. A light bulb is a sphere with radius $1/2$ in. with the bottom sliced off to fit exactly onto a cylinder of radius $1/4$ in. and length $1/3$ in., as seen here. The sphere is cut off at the bottom to fit exactly onto the cylinder, so the radius of the cut is $1/4$ in. Find the surface area (not including the top or bottom of the cylinder).



209. **[T]** A lampshade is constructed by rotating $y = 1/x$ around the x-axis from $y = 1$ to $y = 2$, as seen here. Determine how much material you would need to construct this lampshade—that is, the surface area—accurate to four decimal places.

210. **[T]** An anchor drags behind a boat according to the function $y = 24e^{-x/2} - 24$, where y represents the depth beneath the boat and x is the horizontal distance of the anchor from the back of the boat. If the anchor is 23 ft below the boat, how much rope do you have to pull to reach the anchor? Round your answer to three decimal places.

211. **[T]** You are building a bridge that will span 10 ft. You intend to add decorative rope in the shape of $y = 5|\sin((x\pi)/5)|$, where x is the distance in feet from one end of the bridge. Find out how much rope you need to buy, rounded to the nearest foot.

For the following exercises, find the exact arc length for the following problems over the given interval.

212. $y = \ln(\sin x)$ from $x = \pi/4$ to $x = (3\pi)/4$. (*Hint:* Recall trigonometric identities.)

213. Draw graphs of $y = x^2$, $y = x^6$, and $y = x^{10}$. For $y = x^n$, as n increases, formulate a prediction on the arc length from $(0, 0)$ to $(1, 1)$. Now, compute the lengths of these three functions and determine whether your prediction is correct.

214. Compare the lengths of the parabola $x = y^2$ and the line $x = by$ from $(0, 0)$ to $\left(b^2, b\right)$ as b increases. What do you notice?

215. Solve for the length of $x = y^2$ from $(0, 0)$ to $(1, 1)$. Show that $x = (1/2)y^2$ from $(0, 0)$ to $(2, 2)$ is twice as long. Graph both functions and explain why this is so.

216. **[T]** Which is longer between $(1, 1)$ and $(2, 1/2)$: the hyperbola $y = 1/x$ or the graph of $x + 2y = 3$?

217. Explain why the surface area is infinite when $y = 1/x$ is rotated around the x-axis for $1 \le x < \infty$, but the volume is finite.

6.5 | Physical Applications

Learning Objectives

6.5.1 Determine the mass of a one-dimensional object from its linear density function.
6.5.2 Determine the mass of a two-dimensional circular object from its radial density function.
6.5.3 Calculate the work done by a variable force acting along a line.
6.5.4 Calculate the work done in pumping a liquid from one height to another.
6.5.5 Find the hydrostatic force against a submerged vertical plate.

In this section, we examine some physical applications of integration. Let's begin with a look at calculating mass from a density function. We then turn our attention to work, and close the section with a study of hydrostatic force.

Mass and Density

We can use integration to develop a formula for calculating mass based on a density function. First we consider a thin rod or wire. Orient the rod so it aligns with the x-axis, with the left end of the rod at $x = a$ and the right end of the rod at $x = b$ (**Figure 6.48**). Note that although we depict the rod with some thickness in the figures, for mathematical purposes we assume the rod is thin enough to be treated as a one-dimensional object.

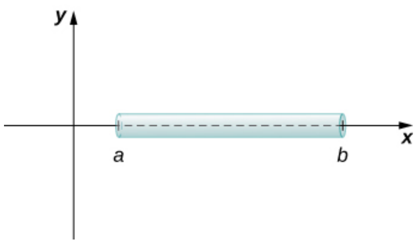


Figure 6.48 We can calculate the mass of a thin rod oriented along the x-axis by integrating its density function.

If the rod has constant density ρ, given in terms of mass per unit length, then the mass of the rod is just the product of the density and the length of the rod: $(b - a)\rho$. If the density of the rod is not constant, however, the problem becomes a little more challenging. When the density of the rod varies from point to point, we use a linear **density function**, $\rho(x)$, to denote the density of the rod at any point, x. Let $\rho(x)$ be an integrable linear density function. Now, for $i = 0, 1, 2,\ldots, n$ let $P = \{x_i\}$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2,\ldots, n$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. **Figure 6.49** shows a representative segment of the rod.

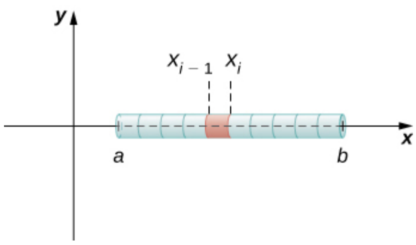


Figure 6.49 A representative segment of the rod.

The mass m_i of the segment of the rod from x_{i-1} to x_i is approximated by

$$m_i \approx \rho(x_i^*)(x_i - x_{i-1}) = \rho(x_i^*)\Delta x.$$

Adding the masses of all the segments gives us an approximation for the mass of the entire rod:

$$m = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n} \rho(x_i^*)\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we get an expression for the exact mass of the rod:

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho(x_i^*)\Delta x = \int_a^b \rho(x)dx.$$

We state this result in the following theorem.

Theorem 6.7: Mass–Density Formula of a One-Dimensional Object

Given a thin rod oriented along the x-axis over the interval $[a, b]$, let $\rho(x)$ denote a linear density function giving the density of the rod at a point x in the interval. Then the mass of the rod is given by

$$m = \int_a^b \rho(x)dx. \tag{6.10}$$

We apply this theorem in the next example.

Example 6.23

Calculating Mass from Linear Density

Consider a thin rod oriented on the x-axis over the interval $[\pi/2, \pi]$. If the density of the rod is given by $\rho(x) = \sin x$, what is the mass of the rod?

Solution

Applying **Equation 6.10** directly, we have

$$m = \int_a^b \rho(x)dx = \int_{\pi/2}^{\pi} \sin x\, dx = -\cos x|_{\pi/2}^{\pi} = 1.$$

6.23 Consider a thin rod oriented on the x-axis over the interval $[1, 3]$. If the density of the rod is given by $\rho(x) = 2x^2 + 3$, what is the mass of the rod?

We now extend this concept to find the mass of a two-dimensional disk of radius r. As with the rod we looked at in the one-dimensional case, here we assume the disk is thin enough that, for mathematical purposes, we can treat it as a two-dimensional object. We assume the density is given in terms of mass per unit area (called *area density*), and further assume the density varies only along the disk's radius (called *radial density*). We orient the disk in the xy-plane, with the center at the origin. Then, the density of the disk can be treated as a function of x, denoted $\rho(x)$. We assume $\rho(x)$ is integrable. Because density is a function of x, we partition the interval from $[0, r]$ along the x-axis. For $i = 0, 1, 2,\ldots, n$, let $P = \{x_i\}$ be a regular partition of the interval $[0, r]$, and for $i = 1, 2,\ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Now, use the partition to break up the disk into thin (two-dimensional) washers. A disk and a representative washer are depicted in the following figure.

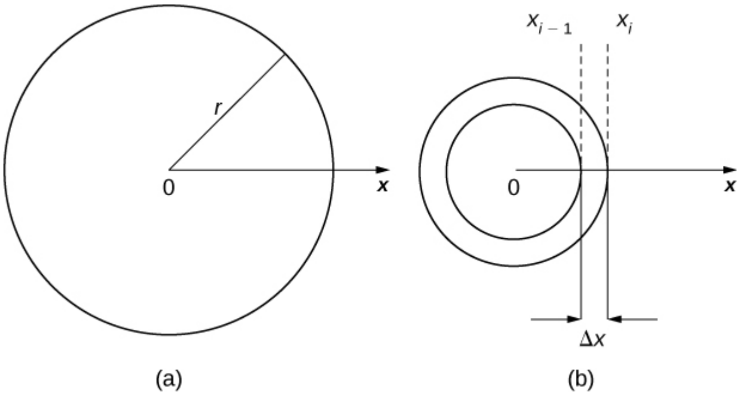


Figure 6.50 (a) A thin disk in the *xy*-plane. (b) A representative washer.

We now approximate the density and area of the washer to calculate an approximate mass, m_i. Note that the area of the washer is given by

$$\begin{aligned} A_i &= \pi(x_i)^2 - \pi(x_{i-1})^2 \\ &= \pi\left[x_i^2 - x_{i-1}^2\right] \\ &= \pi(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \pi(x_i + x_{i-1})\Delta x. \end{aligned}$$

You may recall that we had an expression similar to this when we were computing volumes by shells. As we did there, we use $x_i^* \approx (x_i + x_{i-1})/2$ to approximate the average radius of the washer. We obtain

$$A_i = \pi(x_i + x_{i-1})\Delta x \approx 2\pi x_i^* \Delta x.$$

Using $\rho(x_i^*)$ to approximate the density of the washer, we approximate the mass of the washer by

$$m_i \approx 2\pi x_i^* \rho(x_i^*)\Delta x.$$

Adding up the masses of the washers, we see the mass m of the entire disk is approximated by

$$m = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x.$$

We again recognize this as a Riemann sum, and take the limit as $n \to \infty$. This gives us

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x = \int_0^r 2\pi x \rho(x) dx.$$

We summarize these findings in the following theorem.

Theorem 6.8: Mass–Density Formula of a Circular Object

Let $\rho(x)$ be an integrable function representing the radial density of a disk of radius r. Then the mass of the disk is given by

$$m = \int_0^r 2\pi x \rho(x) dx. \quad (6.11)$$

Example 6.24

Calculating Mass from Radial Density

Let $\rho(x) = \sqrt{x}$ represent the radial density of a disk. Calculate the mass of a disk of radius 4.

Solution

Applying the formula, we find

$$\begin{aligned} m &= \int_0^r 2\pi x\rho(x)dx \\ &= \int_0^4 2\pi x\sqrt{x}dx = 2\pi\int_0^4 x^{3/2}dx \\ &= 2\pi\frac{2}{5}x^{5/2}\Big|_0^4 = \frac{4\pi}{5}[32] = \frac{128\pi}{5}. \end{aligned}$$

6.24 Let $\rho(x) = 3x + 2$ represent the radial density of a disk. Calculate the mass of a disk of radius 2.

Work Done by a Force

We now consider work. In physics, work is related to force, which is often intuitively defined as a push or pull on an object. When a force moves an object, we say the force does work on the object. In other words, work can be thought of as the amount of energy it takes to move an object. According to physics, when we have a constant force, work can be expressed as the product of force and distance.

In the English system, the unit of force is the pound and the unit of distance is the foot, so work is given in foot-pounds. In the metric system, kilograms and meters are used. One newton is the force needed to accelerate 1 kilogram of mass at the rate of 1 m/sec^2. Thus, the most common unit of work is the newton-meter. This same unit is also called the *joule*. Both are defined as kilograms times meters squared over seconds squared $\left(\text{kg}\cdot\text{m}^2/\text{s}^2\right)$.

When we have a constant force, things are pretty easy. It is rare, however, for a force to be constant. The work done to compress (or elongate) a spring, for example, varies depending on how far the spring has already been compressed (or stretched). We look at springs in more detail later in this section.

Suppose we have a variable force $F(x)$ that moves an object in a positive direction along the x-axis from point a to point b. To calculate the work done, we partition the interval $[a, b]$ and estimate the work done over each subinterval. So, for $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of the interval $[a, b],$ and for $i = 1, 2,\ldots, n,$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i].$ To calculate the work done to move an object from point x_{i-1} to point $x_i,$ we assume the force is roughly constant over the interval, and use $F(x_i^*)$ to approximate the force. The work done over the interval $[x_{i-1}, x_i],$ then, is given by

$$W_i \approx F(x_i^*)(x_i - x_{i-1}) = F(x_i^*)\Delta x.$$

Therefore, the work done over the interval $[a, b]$ is approximately

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} F(x_i^*)\Delta x.$$

Taking the limit of this expression as $n \to \infty$ gives us the exact value for work:

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} F(x_i^*)\Delta x = \int_a^b F(x)dx.$$

Thus, we can define work as follows.

Definition

If a variable force $F(x)$ moves an object in a positive direction along the x-axis from point a to point b, then the **work** done on the object is

$$W = \int_a^b F(x)dx. \tag{6.12}$$

Note that if F is constant, the integral evaluates to $F \cdot (b-a) = F \cdot d,$ which is the formula we stated at the beginning of this section.

Now let’s look at the specific example of the work done to compress or elongate a spring. Consider a block attached to a horizontal spring. The block moves back and forth as the spring stretches and compresses. Although in the real world we would have to account for the force of friction between the block and the surface on which it is resting, we ignore friction here and assume the block is resting on a frictionless surface. When the spring is at its natural length (at rest), the system is said to be at equilibrium. In this state, the spring is neither elongated nor compressed, and in this equilibrium position the block does not move until some force is introduced. We orient the system such that $x = 0$ corresponds to the equilibrium position (see the following figure).

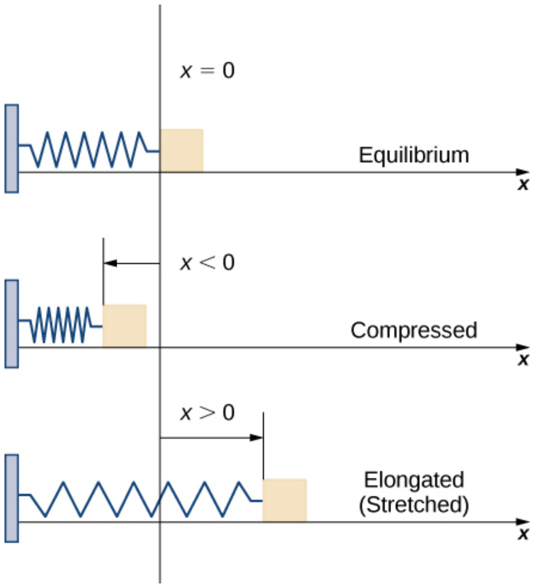


Figure 6.51 A block attached to a horizontal spring at equilibrium, compressed, and elongated.

According to **Hooke’s law**, the force required to compress or stretch a spring from an equilibrium position is given by $F(x) = kx,$ for some constant k. The value of k depends on the physical characteristics of the spring. The constant k is called the *spring constant* and is always positive. We can use this information to calculate the work done to compress or elongate a spring, as shown in the following example.

Example 6.25

The Work Required to Stretch or Compress a Spring

Suppose it takes a force of 10 N (in the negative direction) to compress a spring 0.2 m from the equilibrium position. How much work is done to stretch the spring 0.5 m from the equilibrium position?

Solution

First find the spring constant, k. When $x = -0.2,$ we know $F(x) = -10,$ so

$$\begin{aligned} F(x) &= kx \\ -10 &= k(-0.2) \\ k &= 50 \end{aligned}$$

and $F(x) = 50x.$ Then, to calculate work, we integrate the force function, obtaining

$$W = \int_a^b F(x)dx = \int_0^{0.5} 50x\,dx = 25x^2\Big|_0^{0.5} = 6.25.$$

The work done to stretch the spring is 6.25 J.

6.25 Suppose it takes a force of 8 lb to stretch a spring 6 in. from the equilibrium position. How much work is done to stretch the spring 1 ft from the equilibrium position?

Work Done in Pumping

Consider the work done to pump water (or some other liquid) out of a tank. Pumping problems are a little more complicated than spring problems because many of the calculations depend on the shape and size of the tank. In addition, instead of being concerned about the work done to move a single mass, we are looking at the work done to move a volume of water, and it takes more work to move the water from the bottom of the tank than it does to move the water from the top of the tank.

We examine the process in the context of a cylindrical tank, then look at a couple of examples using tanks of different shapes. Assume a cylindrical tank of radius 4 m and height 10 m is filled to a depth of 8 m. How much work does it take to pump all the water over the top edge of the tank?

The first thing we need to do is define a frame of reference. We let x represent the vertical distance below the top of the tank. That is, we orient the $x\text{-axis}$ vertically, with the origin at the top of the tank and the downward direction being positive (see the following figure).

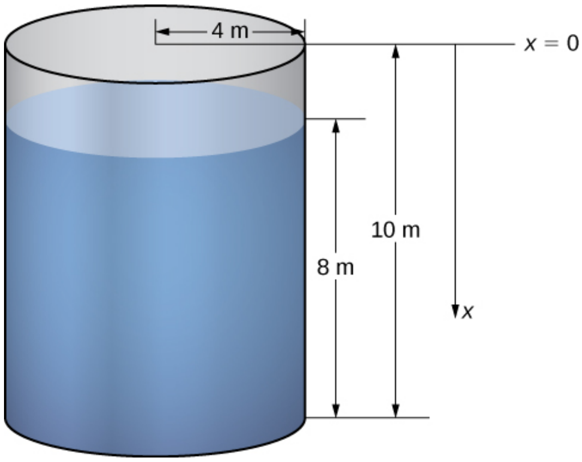


Figure 6.52 How much work is needed to empty a tank partially filled with water?

Using this coordinate system, the water extends from $x = 2$ to $x = 10$. Therefore, we partition the interval $[2, \ 10]$ and look at the work required to lift each individual "layer" of water. So, for $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of the interval $[2, \ 10],$ and for $i = 1, 2,\ldots, n,$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i].$ Figure 6.53 shows a representative layer.

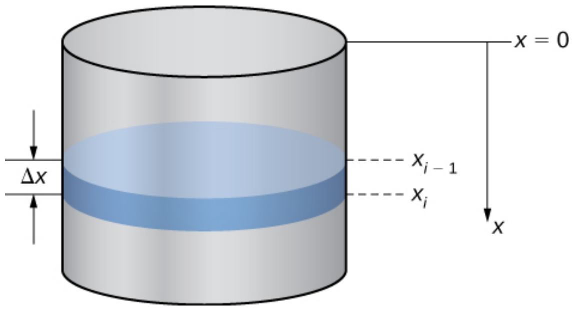


Figure 6.53 A representative layer of water.

In pumping problems, the force required to lift the water to the top of the tank is the force required to overcome gravity, so it is equal to the weight of the water. Given that the weight-density of water is $9800 \ \text{N/m}^3$, or $62.4 \ \text{lb/ft}^3$, calculating the volume of each layer gives us the weight. In this case, we have

$$V = \pi(4)^2 \Delta x = 16\pi\Delta x.$$

Then, the force needed to lift each layer is

$$F = 9800 \cdot 16\pi\Delta x = 156{,}800\pi\Delta x.$$

Note that this step becomes a little more difficult if we have a noncylindrical tank. We look at a noncylindrical tank in the next example.

We also need to know the distance the water must be lifted. Based on our choice of coordinate systems, we can use x_i^* as an approximation of the distance the layer must be lifted. Then the work to lift the ith layer of water W_i is approximately

$$W_i \approx 156{,}800\pi x_i^* \Delta x.$$

Adding the work for each layer, we see the approximate work to empty the tank is given by

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 156{,}800\pi x_i^* \ \Delta x.$$

This is a Riemann sum, so taking the limit as $n \to \infty$, we get

$$\begin{aligned} W &= \lim_{n \to \infty} \sum_{i=1}^{n} 156{,}800\pi x_i^* \ \Delta x \\ &= 156{,}800\pi \int_{2}^{10} x dx \\ &= 156{,}800\pi \left[\frac{x^2}{2}\right]\Bigg|_{2}^{10} = 7{,}526{,}400\pi \approx 23{,}644{,}883. \end{aligned}$$

The work required to empty the tank is approximately 23,650,000 J.

For pumping problems, the calculations vary depending on the shape of the tank or container. The following problem-solving strategy lays out a step-by-step process for solving pumping problems.

Problem-Solving Strategy: Solving Pumping Problems

1. Sketch a picture of the tank and select an appropriate frame of reference.
2. Calculate the volume of a representative layer of water.
3. Multiply the volume by the weight-density of water to get the force.
4. Calculate the distance the layer of water must be lifted.
5. Multiply the force and distance to get an estimate of the work needed to lift the layer of water.
6. Sum the work required to lift all the layers. This expression is an estimate of the work required to pump out the desired amount of water, and it is in the form of a Riemann sum.
7. Take the limit as $n \to \infty$ and evaluate the resulting integral to get the exact work required to pump out the desired amount of water.

We now apply this problem-solving strategy in an example with a noncylindrical tank.

Example 6.26

A Pumping Problem with a Noncylindrical Tank

Assume a tank in the shape of an inverted cone, with height 12 ft and base radius 4 ft. The tank is full to start with, and water is pumped over the upper edge of the tank until the height of the water remaining in the tank is 4 ft. How much work is required to pump out that amount of water?

Solution

The tank is depicted in **Figure 6.54**. As we did in the example with the cylindrical tank, we orient the x-axis vertically, with the origin at the top of the tank and the downward direction being positive (step 1).

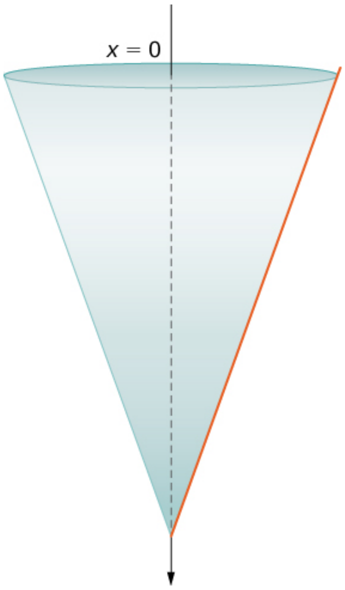


Figure 6.54 A water tank in the shape of an inverted cone.

The tank starts out full and ends with 4 ft of water left, so, based on our chosen frame of reference, we need to partition the interval $[0, 8]$. Then, for $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of the interval $[0, 8],$ and for $i = 1, 2,\ldots, n,$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. We can approximate the volume of a layer by using a disk, then use similar triangles to find the radius of the disk (see the following figure).

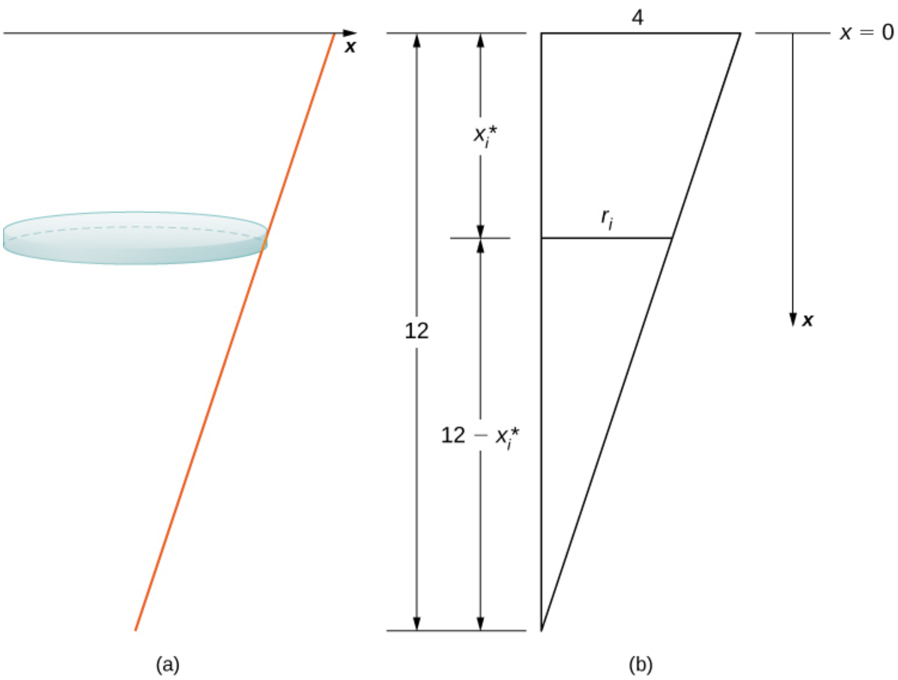


Figure 6.55 Using similar triangles to express the radius of a disk of water.

From properties of similar triangles, we have

$$\begin{aligned}\frac{r_i}{12 - x_i^*} &= \frac{4}{12} = \frac{1}{3} \\ 3r_i &= 12 - x_i^* \\ r_i &= \frac{12 - x_i^*}{3} \\ &= 4 - \frac{x_i^*}{3}.\end{aligned}$$

Then the volume of the disk is

$$V_i = \pi\left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \text{ (step 2)}.$$

The weight-density of water is 62.4 lb/ft^3, so the force needed to lift each layer is approximately

$$F_i \approx 62.4\pi\left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \text{ (step 3)}.$$

Based on the diagram, the distance the water must be lifted is approximately x_i^* feet (step 4), so the approximate work needed to lift the layer is

$$W_i \approx 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \text{ (step 5)}.$$

Summing the work required to lift all the layers, we get an approximate value of the total work:

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \text{ (step 6)}.$$

Taking the limit as $n \to \infty,$ we obtain

$$\begin{aligned}W &= \lim_{n \to \infty} \sum_{i=1}^{n} 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \\ &= \int_0^8 62.4\pi x\left(4 - \frac{x}{3}\right)^2 dx \\ &= 62.4\pi \int_0^8 x\left(16 - \frac{8x}{3} + \frac{x^2}{9}\right)dx = 62.4\pi \int_0^8 \left(16x - \frac{8x^2}{3} + \frac{x^3}{9}\right)dx \\ &= 62.4\pi\left[8x^2 - \frac{8x^3}{9} + \frac{x^4}{36}\right]\Bigg|_0^8 = 10{,}649.6\pi \approx 33{,}456.7.\end{aligned}$$

It takes approximately $33{,}450$ ft-lb of work to empty the tank to the desired level.

6.26 A tank is in the shape of an inverted cone, with height 10 ft and base radius 6 ft. The tank is filled to a depth of 8 ft to start with, and water is pumped over the upper edge of the tank until 3 ft of water remain in the tank. How much work is required to pump out that amount of water?

Hydrostatic Force and Pressure

In this last section, we look at the force and pressure exerted on an object submerged in a liquid. In the English system, force is measured in pounds. In the metric system, it is measured in newtons. Pressure is force per unit area, so in the English system we have pounds per square foot (or, perhaps more commonly, pounds per square inch, denoted psi). In the metric system we have newtons per square meter, also called *pascals*.

Let's begin with the simple case of a plate of area A submerged horizontally in water at a depth *s* (**Figure 6.56**). Then, the force exerted on the plate is simply the weight of the water above it, which is given by $F = \rho As,$ where ρ is the weight density of water (weight per unit volume). To find the **hydrostatic pressure**—that is, the pressure exerted by water on a submerged object—we divide the force by the area. So the pressure is $p = F/A = \rho s.$

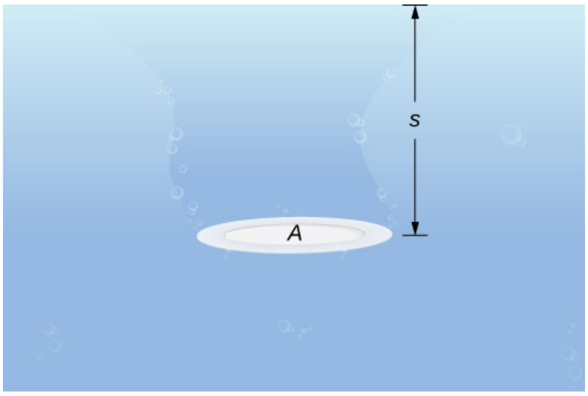


Figure 6.56 A plate submerged horizontally in water.

By Pascal's principle, the pressure at a given depth is the same in all directions, so it does not matter if the plate is submerged horizontally or vertically. So, as long as we know the depth, we know the pressure. We can apply Pascal's principle to find the force exerted on surfaces, such as dams, that are oriented vertically. We cannot apply the formula $F = \rho As$ directly, because the depth varies from point to point on a vertically oriented surface. So, as we have done many times before, we form a partition, a Riemann sum, and, ultimately, a definite integral to calculate the force.

Suppose a thin plate is submerged in water. We choose our frame of reference such that the *x*-axis is oriented vertically, with the downward direction being positive, and point $x = 0$ corresponding to a logical reference point. Let $s(x)$ denote the depth at point *x*. Note we often let $x = 0$ correspond to the surface of the water. In this case, depth at any point is simply given by $s(x) = x.$ However, in some cases we may want to select a different reference point for $x = 0,$ so we proceed with the development in the more general case. Last, let $w(x)$ denote the width of the plate at the point $x.$

Assume the top edge of the plate is at point $x = a$ and the bottom edge of the plate is at point $x = b.$ Then, for $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of the interval $[a, b],$ and for $i = 1, 2,\ldots, n,$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i].$ The partition divides the plate into several thin, rectangular strips (see the following figure).

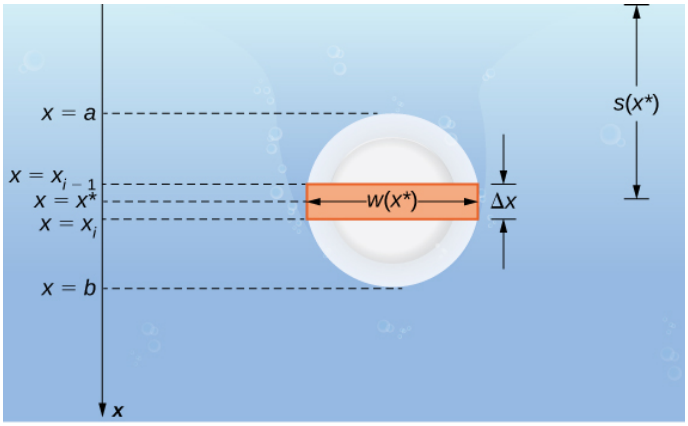


Figure 6.57 A thin plate submerged vertically in water.

Let's now estimate the force on a representative strip. If the strip is thin enough, we can treat it as if it is at a constant depth, $s(x_i^*)$. We then have

$$F_i = \rho As = \rho\left[w(x_i^*)\Delta x\right]s(x_i^*).$$

Adding the forces, we get an estimate for the force on the plate:

$$F \approx \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} \rho\left[w(x_i^*)\Delta x\right]s(x_i^*).$$

This is a Riemann sum, so taking the limit gives us the exact force. We obtain

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho\left[w(x_i^*)\Delta x\right]s(x_i^*) = \int_a^b \rho w(x)s(x)dx. \tag{6.13}$$

Evaluating this integral gives us the force on the plate. We summarize this in the following problem-solving strategy.

Problem-Solving Strategy: Finding Hydrostatic Force

1. Sketch a picture and select an appropriate frame of reference. (Note that if we select a frame of reference other than the one used earlier, we may have to adjust Equation 6.13 accordingly.)
2. Determine the depth and width functions, $s(x)$ and $w(x)$.
3. Determine the weight-density of whatever liquid with which you are working. The weight-density of water is $62.4\ \text{lb/ft}^3$, or 9800 N/m^3.
4. Use the equation to calculate the total force.

Example 6.27

Finding Hydrostatic Force

A water trough 15 ft long has ends shaped like inverted isosceles triangles, with base 8 ft and height 3 ft. Find the force on one end of the trough if the trough is full of water.

Solution

Figure 6.58 shows the trough and a more detailed view of one end.

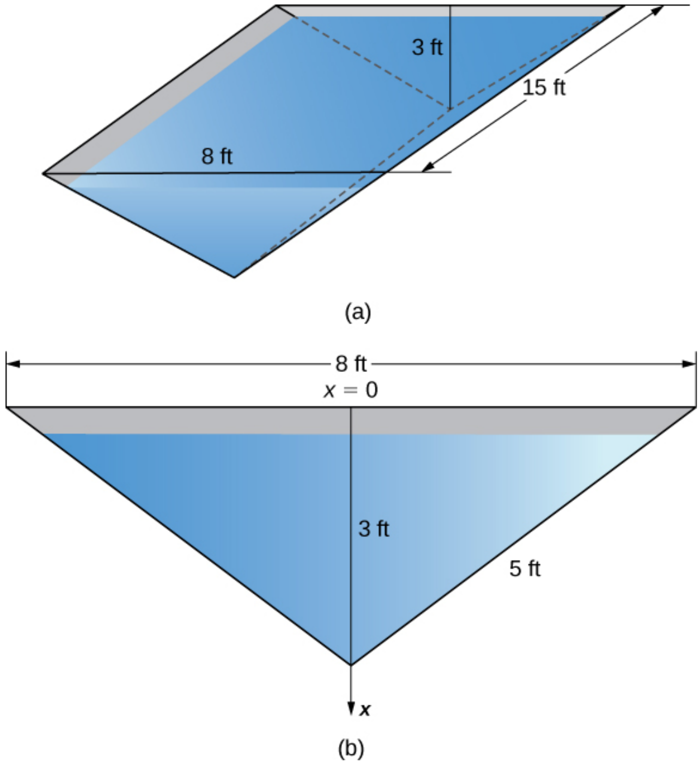


Figure 6.58 (a) A water trough with a triangular cross-section. (b) Dimensions of one end of the water trough.

Select a frame of reference with the x-axis oriented vertically and the downward direction being positive. Select the top of the trough as the point corresponding to $x = 0$ (step 1). The depth function, then, is $s(x) = x$. Using similar triangles, we see that $w(x) = 8 - (8/3)x$ (step 2). Now, the weight density of water is 62.4 lb/ft^3 (step 3), so applying **Equation 6.13**, we obtain

$$\begin{aligned} F &= \int_a^b \rho w(x)s(x)dx \\ &= \int_0^3 62.4\left(8 - \frac{8}{3}x\right)x\,dx = 62.4\int_0^3 \left(8x - \frac{8}{3}x^2\right)dx \\ &= 62.4\left[4x^2 - \frac{8}{9}x^3\right]\Big|_0^3 = 748.8. \end{aligned}$$

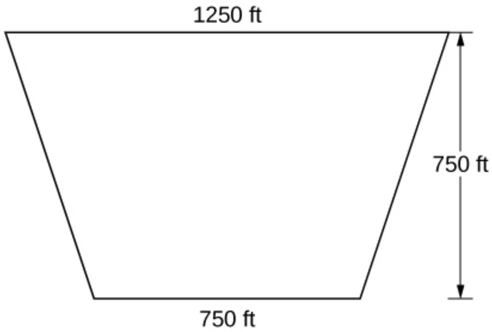
The water exerts a force of 748.8 lb on the end of the trough (step 4).

6.27 A water trough 12 m long has ends shaped like inverted isosceles triangles, with base 6 m and height 4 m. Find the force on one end of the trough if the trough is full of water.

Example 6.28

Chapter Opener: Finding Hydrostatic Force

We now return our attention to the Hoover Dam, mentioned at the beginning of this chapter. The actual dam is arched, rather than flat, but we are going to make some simplifying assumptions to help us with the calculations. Assume the face of the Hoover Dam is shaped like an isosceles trapezoid with lower base 750 ft, upper base 1250 ft, and height 750 ft (see the following figure).



When the reservoir is full, Lake Mead's maximum depth is about 530 ft, and the surface of the lake is about 10 ft below the top of the dam (see the following figure).

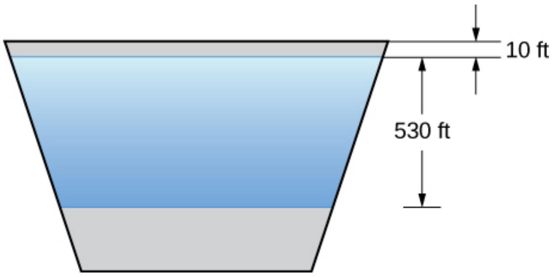


Figure 6.59 A simplified model of the Hoover Dam with assumed dimensions.

a. Find the force on the face of the dam when the reservoir is full.

b. The southwest United States has been experiencing a drought, and the surface of Lake Mead is about 125 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

Solution

a. We begin by establishing a frame of reference. As usual, we choose to orient the x-axis vertically, with the downward direction being positive. This time, however, we are going to let $x = 0$ represent the top of the dam, rather than the surface of the water. When the reservoir is full, the surface of the water is 10 ft below the top of the dam, so $s(x) = x - 10$ (see the following figure).

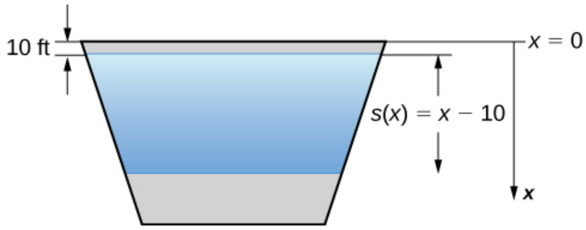


Figure 6.60 We first choose a frame of reference.

To find the width function, we again turn to similar triangles as shown in the figure below.

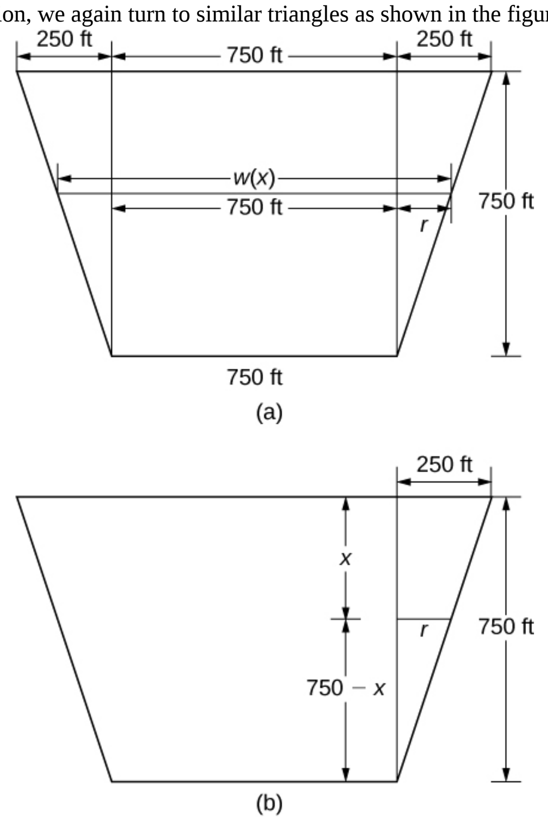


Figure 6.61 We use similar triangles to determine a function for the width of the dam. (a) Assumed dimensions of the dam; (b) highlighting the similar triangles.

From the figure, we see that $w(x) = 750 + 2r$. Using properties of similar triangles, we get $r = 250 - (1/3)x$. Thus,

$$w(x) = 1250 - \frac{2}{3}x \text{ (step 2).}$$

Using a weight-density of 62.4 lb/ft^3 (step 3) and applying **Equation 6.13**, we get

$$\begin{aligned} F &= \int_a^b \rho w(x)s(x)dx \\ &= \int_{10}^{540} 62.4\left(1250 - \frac{2}{3}x\right)(x - 10)dx = 62.4\int_{10}^{540} -\frac{2}{3}\left[x^2 - 1885x + 18750\right]dx \\ &= -62.4\left(\frac{2}{3}\right)\left[\frac{x^3}{3} - \frac{1885x^2}{2} + 18750x\right]\Bigg|_{10}^{540} \approx 8{,}832{,}245{,}000 \text{ lb} = 4{,}416{,}122.5 \text{ t.} \end{aligned}$$

Note the change from pounds to tons $(2000 \text{ lb} = 1 \text{ ton})$ (step 4).

b. Notice that the drought changes our depth function, $s(x)$, and our limits of integration. We have $s(x) = x - 135$. The lower limit of integration is 135. The upper limit remains 540. Evaluating the integral, we get

$$\begin{aligned} F &= \int_a^b \rho w(x)s(x)dx \\ &= \int_{135}^{540} 62.4\left(1250 - \frac{2}{3}x\right)(x - 135)dx \\ &= -62.4\left(\frac{2}{3}\right)\int_{135}^{540} (x - 1875)(x - 135)dx = -62.4\left(\frac{2}{3}\right)\int_{135}^{540} \left(x^2 - 2010x + 253125\right)dx \\ &= -62.4\left(\frac{2}{3}\right)\left[\frac{x^3}{3} - 1005x^2 + 253125x\right]\Bigg|_{135}^{540} \approx 5{,}015{,}230{,}000 \text{ lb} = 2{,}507{,}615 \text{ t.} \end{aligned}$$

6.28 When the reservoir is at its average level, the surface of the water is about 50 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

To learn more about Hoover Dam, see this **article (http://www.openstax.org/l/20_HooverDam)** published by the History Channel.

6.5 EXERCISES

For the following exercises, find the work done.

218. Find the work done when a constant force $F = 12$ lb moves a chair from $x = 0.9$ to $x = 1.1$ ft.

219. How much work is done when a person lifts a 50 lb box of comics onto a truck that is 3 ft off the ground?

220. What is the work done lifting a 20 kg child from the floor to a height of 2 m? (Note that 1 kg equates to 9.8 N)

221. Find the work done when you push a box along the floor 2 m, when you apply a constant force of $F = 100\,\text{N}$.

222. Compute the work done for a force $F = 12/x^2$ N from $x = 1$ to $x = 2$ m.

223. What is the work done moving a particle from $x = 0$ to $x = 1$ m if the force acting on it is $F = 3x^2$ N?

For the following exercises, find the mass of the one-dimensional object.

224. A wire that is 2 ft long (starting at $x = 0$) and has a density function of $\rho(x) = x^2 + 2x$ lb/ft

225. A car antenna that is 3 ft long (starting at $x = 0$) and has a density function of $\rho(x) = 3x + 2$ lb/ft

226. A metal rod that is 8 in. long (starting at $x = 0$) and has a density function of $\rho(x) = e^{1/2x}$ lb/in.

227. A pencil that is 4 in. long (starting at $x = 2$) and has a density function of $\rho(x) = 5/x$ oz/in.

228. A ruler that is 12 in. long (starting at $x = 5$) and has a density function of $\rho(x) = \ln(x) + (1/2)x^2$ oz/in.

For the following exercises, find the mass of the two-dimensional object that is centered at the origin.

229. An oversized hockey puck of radius 2 in. with density function $\rho(x) = x^3 - 2x + 5$

230. A frisbee of radius 6 in. with density function $\rho(x) = e^{-x}$

231. A plate of radius 10 in. with density function $\rho(x) = 1 + \cos(\pi x)$

232. A jar lid of radius 3 in. with density function $\rho(x) = \ln(x + 1)$

233. A disk of radius 5 cm with density function $\rho(x) = \sqrt{3x}$

234. A 12-in. spring is stretched to 15 in. by a force of 75 lb. What is the spring constant?

235. A spring has a natural length of 10 cm. It takes 2 J to stretch the spring to 15 cm. How much work would it take to stretch the spring from 15 cm to 20 cm?

236. A 1-m spring requires 10 J to stretch the spring to 1.1 m. How much work would it take to stretch the spring from 1 m to 1.2 m?

237. A spring requires 5 J to stretch the spring from 8 cm to 12 cm, and an additional 4 J to stretch the spring from 12 cm to 14 cm. What is the natural length of the spring?

238. A shock absorber is compressed 1 in. by a weight of 1 t. What is the spring constant?

239. A force of $F = 20x - x^3$ N stretches a nonlinear spring by x meters. What work is required to stretch the spring from $x = 0$ to $x = 2$ m?

240. Find the work done by winding up a hanging cable of length 100 ft and weight-density 5 lb/ft.

241. For the cable in the preceding exercise, how much work is done to lift the cable 50 ft?

242. For the cable in the preceding exercise, how much additional work is done by hanging a 200 lb weight at the end of the cable?

243. **[T]** A pyramid of height 500 ft has a square base 800 ft by 800 ft. Find the area A at height h. If the rock used to build the pyramid weighs approximately $w = 100\,\text{lb/ft}^3$, how much work did it take to lift all the rock?

244. **[T]** For the pyramid in the preceding exercise, assume there were 1000 workers each working 10 hours a day, 5 days a week, 50 weeks a year. If the workers, on average, lifted 10 100 lb rocks 2 ft/hr, how long did it take to build the pyramid?

245. **[T]** The force of gravity on a mass m is $F = -\left((GMm)/x^2\right)$ newtons. For a rocket of mass $m = 1000\,\text{kg},$ compute the work to lift the rocket from $x = 6400$ to $x = 6500$ m. State your answers with three significant figures. (*Note*: $G = 6.67 \times 10^{-17}$ N m^2/kg^2 and $M = 6 \times 10^{24}$ kg.)

246. **[T]** For the rocket in the preceding exercise, find the work to lift the rocket from $x = 6400$ to $x = \infty$.

247. **[T]** A rectangular dam is 40 ft high and 60 ft wide. Compute the total force F on the dam when

a. the surface of the water is at the top of the dam and
b. the surface of the water is halfway down the dam.

248. **[T]** Find the work required to pump all the water out of a cylinder that has a circular base of radius 5 ft and height 200 ft. Use the fact that the density of water is 62 lb/ft^3.

249. **[T]** Find the work required to pump all the water out of the cylinder in the preceding exercise if the cylinder is only half full.

250. **[T]** How much work is required to pump out a swimming pool if the area of the base is 800 ft^2, the water is 4 ft deep, and the top is 1 ft above the water level? Assume that the density of water is 62 lb/ft^3.

251. A cylinder of depth H and cross-sectional area A stands full of water at density ρ. Compute the work to pump all the water to the top.

252. For the cylinder in the preceding exercise, compute the work to pump all the water to the top if the cylinder is only half full.

253. A cone-shaped tank has a cross-sectional area that increases with its depth: $A = \left(\pi r^2 h^2\right)/H^3$. Show that the work to empty it is half the work for a cylinder with the same height and base.

6.6 | Moments and Centers of Mass

Learning Objectives

6.6.1 Find the center of mass of objects distributed along a line.
6.6.2 Locate the center of mass of a thin plate.
6.6.3 Use symmetry to help locate the centroid of a thin plate.
6.6.4 Apply the theorem of Pappus for volume.

In this section, we consider centers of mass (also called *centroids*, under certain conditions) and moments. The basic idea of the center of mass is the notion of a balancing point. Many of us have seen performers who spin plates on the ends of sticks. The performers try to keep several of them spinning without allowing any of them to drop. If we look at a single plate (without spinning it), there is a sweet spot on the plate where it balances perfectly on the stick. If we put the stick anywhere other than that sweet spot, the plate does not balance and it falls to the ground. (That is why performers spin the plates; the spin helps keep the plates from falling even if the stick is not exactly in the right place.) Mathematically, that sweet spot is called the *center of mass of the plate*.

In this section, we first examine these concepts in a one-dimensional context, then expand our development to consider centers of mass of two-dimensional regions and symmetry. Last, we use centroids to find the volume of certain solids by applying the theorem of Pappus.

Center of Mass and Moments

Let's begin by looking at the center of mass in a one-dimensional context. Consider a long, thin wire or rod of negligible mass resting on a fulcrum, as shown in Figure 6.62(a). Now suppose we place objects having masses m_1 and m_2 at distances d_1 and d_2 from the fulcrum, respectively, as shown in Figure 6.62(b).

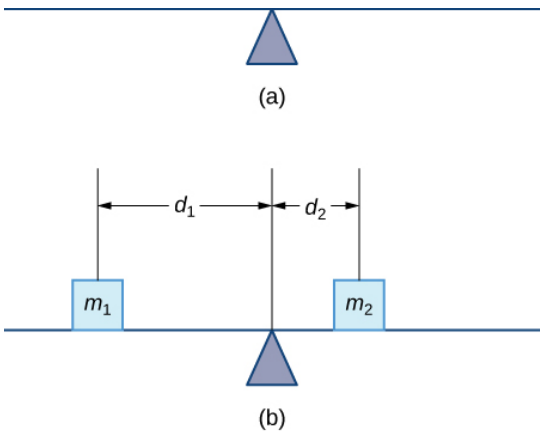


Figure 6.62 (a) A thin rod rests on a fulcrum. (b) Masses are placed on the rod.

The most common real-life example of a system like this is a playground seesaw, or teeter-totter, with children of different weights sitting at different distances from the center. On a seesaw, if one child sits at each end, the heavier child sinks down and the lighter child is lifted into the air. If the heavier child slides in toward the center, though, the seesaw balances. Applying this concept to the masses on the rod, we note that the masses balance each other if and only if $m_1 d_1 = m_2 d_2$.

In the seesaw example, we balanced the system by moving the masses (children) with respect to the fulcrum. However, we are really interested in systems in which the masses are not allowed to move, and instead we balance the system by moving the fulcrum. Suppose we have two point masses, m_1 and m_2, located on a number line at points x_1 and x_2, respectively (Figure 6.63). The center of mass, $\bar{x}$, is the point where the fulcrum should be placed to make the system balance.

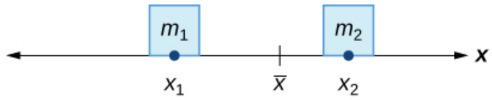


Figure 6.63 The center of mass $\bar{x}$ is the balance point of the system.

Thus, we have

$$\begin{aligned}
m_1|x_1 - \bar{x}| &= m_2|x_2 - \bar{x}| \\
m_1(\bar{x} - x_1) &= m_2(x_2 - \bar{x}) \\
m_1\bar{x} - m_1x_1 &= m_2x_2 - m_2\bar{x} \\
\bar{x}(m_1 + m_2) &= m_1x_1 + m_2x_2 \\
\bar{x} &= \frac{m_1x_1 + m_2x_2}{m_1 + m_2}.
\end{aligned}$$

The expression in the numerator, $m_1x_1 + m_2x_2,$ is called the *first moment of the system with respect to the origin.* If the context is clear, we often drop the word *first* and just refer to this expression as the **moment** of the system. The expression in the denominator, $m_1 + m_2,$ is the total mass of the system. Thus, the **center of mass** of the system is the point at which the total mass of the system could be concentrated without changing the moment.

This idea is not limited just to two point masses. In general, if *n* masses, $m_1, m_2, \ldots, m_n,$ are placed on a number line at points $x_1, x_2, \ldots, x_n,$ respectively, then the center of mass of the system is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}.$$

Theorem 6.9: Center of Mass of Objects on a Line

Let $m_1, m_2, \ldots, m_n$ be point masses placed on a number line at points $x_1, x_2, \ldots, x_n,$ respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then, the moment of the system with respect to the origin is given by

$$M = \sum_{i=1}^{n} m_i x_i \tag{6.14}$$

and the center of mass of the system is given by

$$\bar{x} = \frac{M}{m}. \tag{6.15}$$

We apply this theorem in the following example.

Example 6.29

Finding the Center of Mass of Objects along a Line

Suppose four point masses are placed on a number line as follows:

$$m_1 = 30\text{ kg, placed at } x_1 = -2\text{ m} \qquad m_2 = 5\text{ kg, placed at } x_2 = 3\text{ m}$$
$$m_3 = 10\text{ kg, placed at } x_3 = 6\text{ m} \qquad m_4 = 15\text{ kg, placed at } x_4 = -3\text{ m}.$$

Find the moment of the system with respect to the origin and find the center of mass of the system.

Solution

First, we need to calculate the moment of the system:

$$\begin{aligned} M &= \sum_{i=1}^{4} m_i x_i \\ &= -60 + 15 + 60 - 45 = -30. \end{aligned}$$

Now, to find the center of mass, we need the total mass of the system:

$$\begin{aligned} m &= \sum_{i=1}^{4} m_i \\ &= 30 + 5 + 10 + 15 = 60\text{ kg}. \end{aligned}$$

Then we have

$$\bar{x} = \frac{M}{m} = \frac{-30}{60} = -\frac{1}{2}.$$

The center of mass is located 1/2 m to the left of the origin.

6.29 Suppose four point masses are placed on a number line as follows:

$$m_1 = 12\text{ kg, placed at } x_1 = -4\text{ m} \qquad m_2 = 12\text{ kg, placed at } x_2 = 4\text{ m}$$
$$m_3 = 30\text{ kg, placed at } x_3 = 2\text{ m} \qquad m_4 = 6\text{ kg, placed at } x_4 = -6\text{ m}.$$

Find the moment of the system with respect to the origin and find the center of mass of the system.

We can generalize this concept to find the center of mass of a system of point masses in a plane. Let m_1 be a point mass located at point (x_1, y_1) in the plane. Then the moment M_x of the mass with respect to the *x*-axis is given by $M_x = m_1 y_1$. Similarly, the moment M_y with respect to the *y*-axis is given by $M_y = m_1 x_1$. Notice that the *x*-coordinate of the point is used to calculate the moment with respect to the *y*-axis, and vice versa. The reason is that the *x*-coordinate gives the distance from the point mass to the *y*-axis, and the *y*-coordinate gives the distance to the *x*-axis (see the following figure).

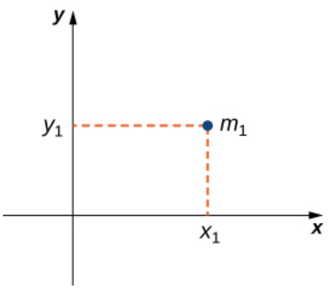


Figure 6.64 Point mass m_1 is located at point (x_1, y_1) in the plane.

If we have several point masses in the *xy*-plane, we can use the moments with respect to the *x*- and *y*-axes to calculate the *x*- and *y*-coordinates of the center of mass of the system.

Theorem 6.10: Center of Mass of Objects in a Plane

Let $m_1, m_2, \ldots, m_n$ be point masses located in the *xy*-plane at points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n),$ respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then the moments M_x and M_y of the system with respect to the *x*- and *y*-axes, respectively, are given by

$$M_x = \sum_{i=1}^{n} m_i y_i \quad \text{and} \quad M_y = \sum_{i=1}^{n} m_i x_i. \tag{6.16}$$

Also, the coordinates of the center of mass $(\bar{x}, \bar{y})$ of the system are

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}. \tag{6.17}$$

The next example demonstrates how to apply this theorem.

Example 6.30

Finding the Center of Mass of Objects in a Plane

Suppose three point masses are placed in the *xy*-plane as follows (assume coordinates are given in meters):

$$\begin{aligned} m_1 &= 2\,\text{kg, placed at } (-1, 3), \\ m_2 &= 6\,\text{kg, placed at } (1, 1), \\ m_3 &= 4\,\text{kg, placed at } (2, -2). \end{aligned}$$

Find the center of mass of the system.

Solution

First we calculate the total mass of the system:

$$m = \sum_{i=1}^{3} m_i = 2 + 6 + 4 = 12\,\text{kg}.$$

Next we find the moments with respect to the *x*- and *y*-axes:

$$\begin{aligned} M_y &= \sum_{i=1}^{3} m_i x_i = -2 + 6 + 8 = 12, \\ M_x &= \sum_{i=1}^{3} m_i y_i = 6 + 6 - 8 = 4. \end{aligned}$$

Then we have

$$\bar{x} = \frac{M_y}{m} = \frac{12}{12} = 1 \text{ and } \bar{y} = \frac{M_x}{m} = \frac{4}{12} = \frac{1}{3}.$$

The center of mass of the system is $(1, 1/3),$ in meters.

6.30 Suppose three point masses are placed on a number line as follows (assume coordinates are given in meters):

$$\begin{aligned} m_1 &= 5 \text{ kg, placed at } (-2,\ -3), \\ m_2 &= 3 \text{ kg, placed at } (2,\ 3), \\ m_3 &= 2 \text{ kg, placed at } (-3,\ -2). \end{aligned}$$

Find the center of mass of the system.

Center of Mass of Thin Plates

So far we have looked at systems of point masses on a line and in a plane. Now, instead of having the mass of a system concentrated at discrete points, we want to look at systems in which the mass of the system is distributed continuously across a thin sheet of material. For our purposes, we assume the sheet is thin enough that it can be treated as if it is two-dimensional. Such a sheet is called a **lamina**. Next we develop techniques to find the center of mass of a lamina. In this section, we also assume the density of the lamina is constant.

Laminas are often represented by a two-dimensional region in a plane. The geometric center of such a region is called its **centroid**. Since we have assumed the density of the lamina is constant, the center of mass of the lamina depends only on the shape of the corresponding region in the plane; it does not depend on the density. In this case, the center of mass of the lamina corresponds to the centroid of the delineated region in the plane. As with systems of point masses, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the x- and y-axes.

We first consider a lamina in the shape of a rectangle. Recall that the center of mass of a lamina is the point where the lamina balances. For a rectangle, that point is both the horizontal and vertical center of the rectangle. Based on this understanding, it is clear that the center of mass of a rectangular lamina is the point where the diagonals intersect, which is a result of the **symmetry principle**, and it is stated here without proof.

Theorem 6.11: The Symmetry Principle

If a region R is symmetric about a line l, then the centroid of R lies on l.

Let's turn to more general laminas. Suppose we have a lamina bounded above by the graph of a continuous function $f(x)$, below by the x-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in the following figure.

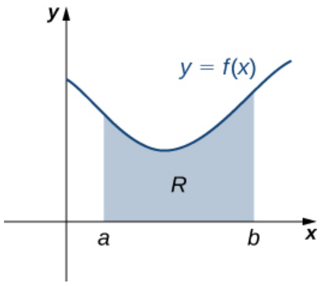


Figure 6.65 A region in the plane representing a lamina.

As with systems of point masses, to find the center of mass of the lamina, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the x- and y-axes. As we have done many times before, we approximate these quantities by partitioning the interval $[a, b]$ and constructing rectangles.

For $i = 0, 1, 2,\ldots, n,$ let $P = \{x_i\}$ be a regular partition of $[a, b]$. Recall that we can choose any point within the interval $[x_{i-1}, x_i]$ as our x_i^*. In this case, we want x_i^* to be the x-coordinate of the centroid of our rectangles. Thus, for $i = 1, 2,\ldots, n,$ we select $x_i^* \in [x_{i-1}, x_i]$ such that x_i^* is the midpoint of the interval. That is, $x_i^* = (x_{i-1} + x_i)/2$. Now, for $i = 1, 2,\ldots, n,$ construct a rectangle of height $f(x_i^*)$ on $[x_{i-1}, x_i]$. The center of mass of this rectangle is

$\left(x_i^*, \left(f(x_i^*)\right)/2\right)$, as shown in the following figure.

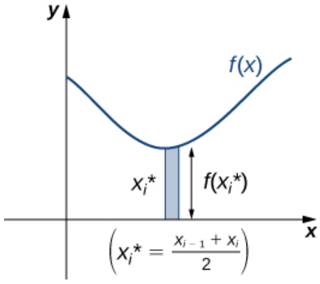


Figure 6.66 A representative rectangle of the lamina.

Next, we need to find the total mass of the rectangle. Let ρ represent the density of the lamina (note that ρ is a constant). In this case, ρ is expressed in terms of mass per unit area. Thus, to find the total mass of the rectangle, we multiply the area of the rectangle by ρ. Then, the mass of the rectangle is given by $\rho f(x_i^*)\Delta x$.

To get the approximate mass of the lamina, we add the masses of all the rectangles to get

$$m \approx \sum_{i=1}^{n} \rho f(x_i^*)\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$ gives the exact mass of the lamina:

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho f(x_i^*)\Delta x = \rho \int_a^b f(x)dx.$$

Next, we calculate the moment of the lamina with respect to the *x*-axis. Returning to the representative rectangle, recall its center of mass is $\left(x_i^*, \left(f(x_i^*)\right)/2\right)$. Recall also that treating the rectangle as if it is a point mass located at the center of mass does not change the moment. Thus, the moment of the rectangle with respect to the *x*-axis is given by the mass of the rectangle, $\rho f(x_i^*)\Delta x$, multiplied by the distance from the center of mass to the *x*-axis: $\left(f(x_i^*)\right)/2$. Therefore, the moment with respect to the *x*-axis of the rectangle is $\rho\left(\left[f(x_i^*)\right]^2/2\right)\Delta x$. Adding the moments of the rectangles and taking the limit of the resulting Riemann sum, we see that the moment of the lamina with respect to the *x*-axis is

$$M_x = \lim_{n \to \infty} \sum_{i=1}^{n} \rho \frac{\left[f(x_i^*)\right]^2}{2}\Delta x = \rho \int_a^b \frac{\left[f(x)\right]^2}{2}dx.$$

We derive the moment with respect to the *y*-axis similarly, noting that the distance from the center of mass of the rectangle to the *y*-axis is x_i^*. Then the moment of the lamina with respect to the *y*-axis is given by

$$M_y = \lim_{n \to \infty} \sum_{i=1}^{n} \rho x_i^* f(x_i^*)\Delta x = \rho \int_a^b x f(x)dx.$$

We find the coordinates of the center of mass by dividing the moments by the total mass to give $\bar{x} = M_y/m$ and $\bar{y} = M_x/m$. If we look closely at the expressions for M_x, M_y, and m, we notice that the constant ρ cancels out when $\bar{x}$ and $\bar{y}$ are calculated.

We summarize these findings in the following theorem.

Theorem 6.12: Center of Mass of a Thin Plate in the *xy*-Plane

Let *R* denote a region bounded above by the graph of a continuous function $f(x)$, below by the *x*-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho \int_a^b f(x)dx. \tag{6.18}$$

ii. The moments M_x and M_y of the lamina with respect to the *x*- and *y*-axes, respectively, are

$$M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx \text{ and } M_y = \rho \int_a^b x f(x)dx. \tag{6.19}$$

iii. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}. \tag{6.20}$$

In the next example, we use this theorem to find the center of mass of a lamina.

Example 6.31

Finding the Center of Mass of a Lamina

Let R be the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the *x*-axis over the interval $[0, 4]$. Find the centroid of the region.

Solution

The region is depicted in the following figure.

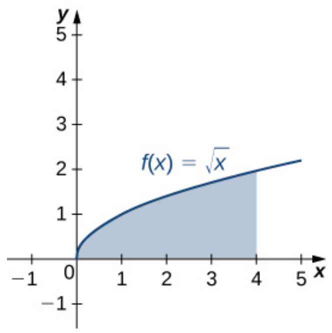


Figure 6.67 Finding the center of mass of a lamina.

Since we are only asked for the centroid of the region, rather than the mass or moments of the associated lamina, we know the density constant ρ cancels out of the calculations eventually. Therefore, for the sake of convenience, let's assume $\rho = 1$.

First, we need to calculate the total mass:

$$\begin{aligned} m &= \rho \int_a^b f(x)dx = \int_0^4 \sqrt{x}\, dx \\ &= \frac{2}{3} x^{3/2} \Big|_0^4 = \frac{2}{3}[8 - 0] = \frac{16}{3}. \end{aligned}$$

Next, we compute the moments:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{[f(x)]^2}{2} dx \\ &= \int_0^4 \frac{x}{2} dx = \frac{1}{4} x^2 \Big|_0^4 = 4 \end{aligned}$$

and

$$\begin{aligned} M_y &= \rho \int_a^b x f(x) dx \\ &= \int_0^4 x\sqrt{x} dx = \int_0^4 x^{3/2} dx \\ &= \frac{2}{5} x^{5/2} \Big|_0^4 = \frac{2}{5}[32 - 0] = \frac{64}{5}. \end{aligned}$$

Thus, we have

$$\bar{x} = \frac{M_y}{m} = \frac{64/5}{16/3} = \frac{64}{5} \cdot \frac{3}{16} = \frac{12}{5} \text{ and } \bar{y} = \frac{M_x}{y} = \frac{4}{16/3} = 4 \cdot \frac{3}{16} = \frac{3}{4}.$$

The centroid of the region is $(12/5, 3/4)$.

6.31 Let R be the region bounded above by the graph of the function $f(x) = x^2$ and below by the x-axis over the interval $[0, 2]$. Find the centroid of the region.

We can adapt this approach to find centroids of more complex regions as well. Suppose our region is bounded above by the graph of a continuous function $f(x)$, as before, but now, instead of having the lower bound for the region be the x-axis, suppose the region is bounded below by the graph of a second continuous function, $g(x)$, as shown in the following figure.

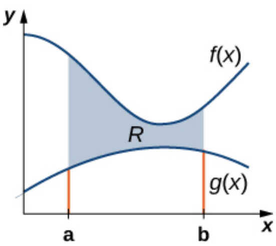


Figure 6.68 A region between two functions.

Again, we partition the interval $[a, b]$ and construct rectangles. A representative rectangle is shown in the following figure.

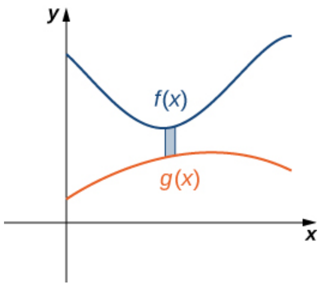


Figure 6.69 A representative rectangle of the region between two functions.

Note that the centroid of this rectangle is $\left(x_i^*, \left(f(x_i^*) + g(x_i^*)\right)/2\right)$. We won't go through all the details of the Riemann sum development, but let's look at some of the key steps. In the development of the formulas for the mass of the lamina and the moment with respect to the y-axis, the height of each rectangle is given by $f(x_i^*) - g(x_i^*)$, which leads to the expression $f(x) - g(x)$ in the integrands.

In the development of the formula for the moment with respect to the x-axis, the moment of each rectangle is found by multiplying the area of the rectangle, $\rho\left[f(x_i^*) - g(x_i^*)\right]\Delta x$, by the distance of the centroid from the x-axis, $\left(f(x_i^*) + g(x_i^*)\right)/2$, which gives $\rho(1/2)\left\{\left[f(x_i^*)\right]^2 - \left[g(x_i^*)\right]^2\right\}\Delta x$. Summarizing these findings, we arrive at the following theorem.

Theorem 6.13: Center of Mass of a Lamina Bounded by Two Functions

Let R denote a region bounded above by the graph of a continuous function $f(x)$, below by the graph of the continuous function $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho\int_a^b \left[f(x) - g(x)\right]dx. \tag{6.21}$$

ii. The moments M_x and M_y of the lamina with respect to the x- and y-axes, respectively, are

$$M_x = \rho\int_a^b \frac{1}{2}\left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)dx \text{ and } M_y = \rho\int_a^b x\left[f(x) - g(x)\right]dx. \tag{6.22}$$

iii. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}. \tag{6.23}$$

We illustrate this theorem in the following example.

Example 6.32

Finding the Centroid of a Region Bounded by Two Functions

Let R be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by the graph of the function $g(x) = x - 1$. Find the centroid of the region.

Solution

The region is depicted in the following figure.

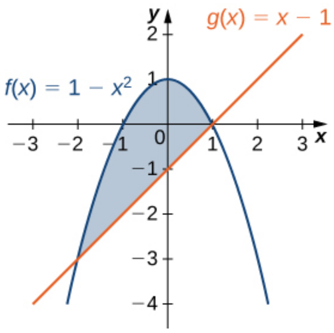


Figure 6.70 Finding the centroid of a region between two curves.

The graphs of the functions intersect at $(-2, -3)$ and $(1, 0)$, so we integrate from −2 to 1. Once again, for the sake of convenience, assume $\rho = 1$.

First, we need to calculate the total mass:

$$\begin{aligned} m &= \rho\int_a^b [f(x) - g(x)]dx \\ &= \int_{-2}^{1}\left[1 - x^2 - (x-1)\right]dx = \int_{-2}^{1}(2 - x^2 - x)dx \\ &= \left[2x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right]\Big|_{-2}^{1} = \left[2 - \frac{1}{3} - \frac{1}{2}\right] - \left[-4 + \frac{8}{3} - 2\right] = \frac{9}{2}. \end{aligned}$$

Next, we compute the moments:

$$\begin{aligned} M_x &= \rho\int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right)dx \\ &= \frac{1}{2}\int_{-2}^{1}\left(\left(1 - x^2\right)^2 - (x-1)^2\right)dx = \frac{1}{2}\int_{-2}^{1}\left(x^4 - 3x^2 + 2x\right)dx \\ &= \frac{1}{2}\left[\frac{x^5}{5} - x^3 + x^2\right]\Big|_{-2}^{1} = -\frac{27}{10} \end{aligned}$$

and

$$\begin{aligned} M_y &= \rho\int_a^b x[f(x) - g(x)]dx \\ &= \int_{-2}^{1} x\left[(1 - x^2) - (x - 1)\right]dx = \int_{-2}^{1} x\left[2 - x^2 - x\right]dx \\ &= \int_{-2}^{1}\left(2x - x^4 - x^2\right)dx \\ &= \left[x^2 - \frac{x^5}{5} - \frac{x^3}{3}\right]\Big|_{-2}^{1} = -\frac{9}{4}. \end{aligned}$$

Therefore, we have

$$\bar{x} = \frac{M_y}{m} = -\frac{9}{4} \cdot \frac{2}{9} = -\frac{1}{2} \text{ and } \bar{y} = \frac{M_x}{m} = -\frac{27}{10} \cdot \frac{2}{9} = -\frac{3}{5}.$$

The centroid of the region is $(-(1/2), -(3/5))$.

6.32 Let R be the region bounded above by the graph of the function $f(x) = 6 - x^2$ and below by the graph of the function $g(x) = 3 - 2x$. Find the centroid of the region.

The Symmetry Principle

We stated the symmetry principle earlier, when we were looking at the centroid of a rectangle. The symmetry principle can be a great help when finding centroids of regions that are symmetric. Consider the following example.

Example 6.33

Finding the Centroid of a Symmetric Region

Let R be the region bounded above by the graph of the function $f(x) = 4 - x^2$ and below by the x-axis. Find the centroid of the region.

Solution

The region is depicted in the following figure.

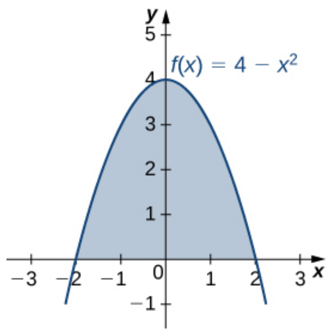


Figure 6.71 We can use the symmetry principle to help find the centroid of a symmetric region.

The region is symmetric with respect to the y-axis. Therefore, the x-coordinate of the centroid is zero. We need only calculate $\bar{y}$. Once again, for the sake of convenience, assume $\rho = 1$.

First, we calculate the total mass:

$$\begin{aligned} m &= \rho\int_a^b f(x)dx \\ &= \int_{-2}^{2}\left(4 - x^2\right)dx \\ &= \left[4x - \frac{x^3}{3}\right]\Bigg|_{-2}^{2} = \frac{32}{3}. \end{aligned}$$

Next, we calculate the moments. We only need M_x:

$$\begin{aligned} M_x &= \rho\int_a^b \frac{[f(x)]^2}{2}dx \\ &= \frac{1}{2}\int_{-2}^{2}\left[4 - x^2\right]^2 dx = \frac{1}{2}\int_{-2}^{2}\left(16 - 8x^2 + x^4\right)dx \\ &= \frac{1}{2}\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]\Bigg|_{-2}^{2} = \frac{256}{15}. \end{aligned}$$

Then we have

$$\bar{y} = \frac{M_x}{m} = \frac{256}{15}\cdot\frac{3}{32} = \frac{8}{5}.$$

The centroid of the region is $(0,\ 8/5)$.

6.33 Let R be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by x-axis. Find the centroid of the region.

Student PROJECT

The Grand Canyon Skywalk

The Grand Canyon Skywalk opened to the public on March 28, 2007. This engineering marvel is a horseshoe-shaped observation platform suspended 4000 ft above the Colorado River on the West Rim of the Grand Canyon. Its crystal-clear glass floor allows stunning views of the canyon below (see the following figure).

Figure 6.72 The Grand Canyon Skywalk offers magnificent views of the canyon. (credit: 10da_ralta, Wikimedia Commons)

The Skywalk is a cantilever design, meaning that the observation platform extends over the rim of the canyon, with no visible means of support below it. Despite the lack of visible support posts or struts, cantilever structures are engineered to be very stable and the Skywalk is no exception. The observation platform is attached firmly to support posts that extend 46 ft down into bedrock. The structure was built to withstand 100-mph winds and an 8.0-magnitude earthquake within 50 mi, and is capable of supporting more than 70,000,000 lb.

One factor affecting the stability of the Skywalk is the center of gravity of the structure. We are going to calculate the center of gravity of the Skywalk, and examine how the center of gravity changes when tourists walk out onto the observation platform.

The observation platform is U-shaped. The legs of the U are 10 ft wide and begin on land, under the visitors' center, 48 ft from the edge of the canyon. The platform extends 70 ft over the edge of the canyon.

To calculate the center of mass of the structure, we treat it as a lamina and use a two-dimensional region in the *xy*-plane to represent the platform. We begin by dividing the region into three subregions so we can consider each subregion separately. The first region, denoted R_1, consists of the curved part of the U. We model R_1 as a semicircular annulus, with inner radius 25 ft and outer radius 35 ft, centered at the origin (see the following figure).

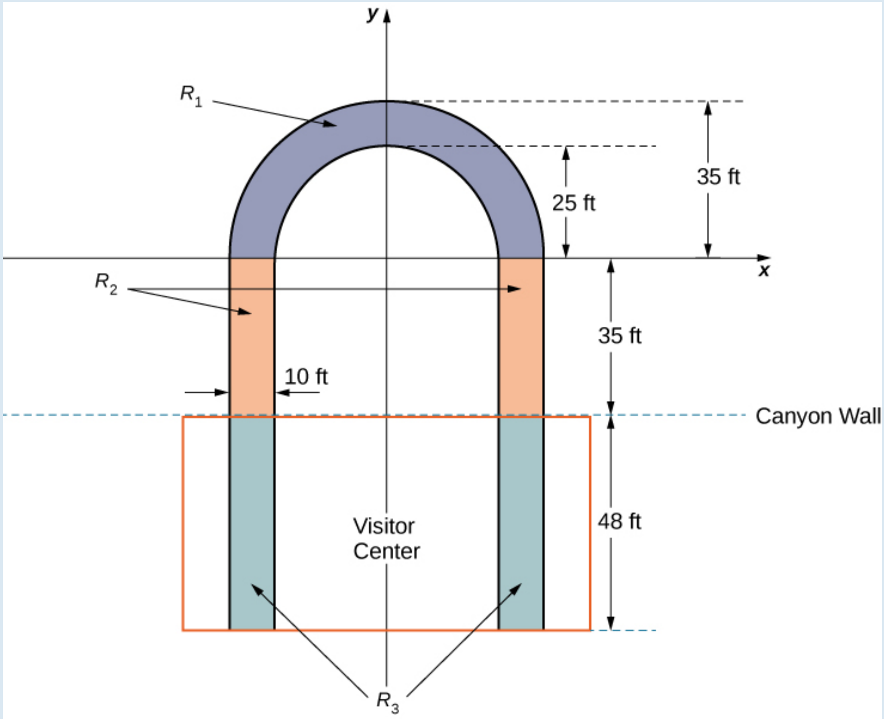


Figure 6.73 We model the Skywalk with three sub-regions.

The legs of the platform, extending 35 ft between R_1 and the canyon wall, comprise the second sub-region, R_2. Last, the ends of the legs, which extend 48 ft under the visitor center, comprise the third sub-region, R_3. Assume the density of the lamina is constant and assume the total weight of the platform is 1,200,000 lb (not including the weight of the visitor center; we will consider that later). Use $g = 32\text{ ft/sec}^2$.

1. Compute the area of each of the three sub-regions. Note that the areas of regions R_2 and R_3 should include the areas of the legs only, not the open space between them. Round answers to the nearest square foot.
2. Determine the mass associated with each of the three sub-regions.
3. Calculate the center of mass of each of the three sub-regions.
4. Now, treat each of the three sub-regions as a point mass located at the center of mass of the corresponding sub-region. Using this representation, calculate the center of mass of the entire platform.
5. Assume the visitor center weighs 2,200,000 lb, with a center of mass corresponding to the center of mass of R_3. Treating the visitor center as a point mass, recalculate the center of mass of the system. How does the center of mass change?
6. Although the Skywalk was built to limit the number of people on the observation platform to 120, the platform is capable of supporting up to 800 people weighing 200 lb each. If all 800 people were allowed on the platform, and all of them went to the farthest end of the platform, how would the center of gravity of the system be affected? (Include the visitor center in the calculations and represent the people by a point mass located at the farthest edge of the platform, 70 ft from the canyon wall.)

Theorem of Pappus

This section ends with a discussion of the **theorem of Pappus for volume**, which allows us to find the volume of particular kinds of solids by using the centroid. (There is also a theorem of Pappus for surface area, but it is much less useful than the theorem for volume.)

Theorem 6.14: Theorem of Pappus for Volume

Let R be a region in the plane and let l be a line in the plane that does not intersect R. Then the volume of the solid of revolution formed by revolving R around l is equal to the area of R multiplied by the distance d traveled by the centroid of R.

Proof

We can prove the case when the region is bounded above by the graph of a function $f(x)$ and below by the graph of a function $g(x)$ over an interval $[a, b]$, and for which the axis of revolution is the y-axis. In this case, the area of the region is $A = \int_a^b [f(x) - g(x)]dx$. Since the axis of rotation is the y-axis, the distance traveled by the centroid of the region depends only on the x-coordinate of the centroid, $\bar{x}$, which is

$$\bar{x} = \frac{M_y}{m},$$

where

$$m = \rho \int_a^b [f(x) - g(x)]dx \text{ and } M_y = \rho \int_a^b x[f(x) - g(x)]dx.$$

Then,

$$d = 2\pi \frac{\rho \int_a^b x[f(x) - g(x)]dx}{\rho \int_a^b [f(x) - g(x)]dx}$$

and thus

$$d \cdot A = 2\pi \int_a^b x[f(x) - g(x)]dx.$$

However, using the method of cylindrical shells, we have

$$V = 2\pi \int_a^b x[f(x) - g(x)]dx.$$

So,

$$V = d \cdot A$$

and the proof is complete.

□

Example 6.34

Using the Theorem of Pappus for Volume

Let R be a circle of radius 2 centered at $(4, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving R around the y-axis.

Solution

The region and torus are depicted in the following figure.

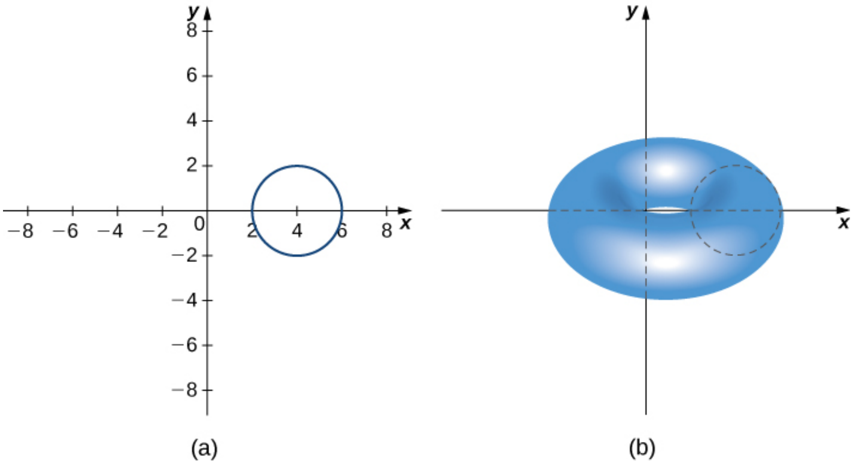


Figure 6.74 Determining the volume of a torus by using the theorem of Pappus. (a) A circular region R in the plane; (b) the torus generated by revolving R about the y-axis.

The region R is a circle of radius 2, so the area of R is $A = 4\pi$ units^2. By the symmetry principle, the centroid of R is the center of the circle. The centroid travels around the y-axis in a circular path of radius 4, so the centroid travels $d = 8\pi$ units. Then, the volume of the torus is $A \cdot d = 32\pi^2$ units^3.

6.34 Let R be a circle of radius 1 centered at $(3, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving R around the y-axis.

6.6 EXERCISES

For the following exercises, calculate the center of mass for the collection of masses given.

254. $m_1 = 2$ at $x_1 = 1$ and $m_2 = 4$ at $x_2 = 2$

255. $m_1 = 1$ at $x_1 = -1$ and $m_2 = 3$ at $x_2 = 2$

256. $m = 3$ at $x = 0, 1, 2, 6$

257. Unit masses at $(x, y) = (1, 0), (0, 1), (1, 1)$

258. $m_1 = 1$ at $(1, 0)$ and $m_2 = 4$ at $(0, 1)$

259. $m_1 = 1$ at $(1, 0)$ and $m_2 = 3$ at $(2, 2)$

For the following exercises, compute the center of mass $\bar{x}$.

260. $\rho = 1$ for $x \in (-1, 3)$

261. $\rho = x^2$ for $x \in (0, L)$

262. $\rho = 1$ for $x \in (0, 1)$ and $\rho = 2$ for $x \in (1, 2)$

263. $\rho = \sin x$ for $x \in (0, \pi)$

264. $\rho = \cos x$ for $x \in \left(0, \frac{\pi}{2}\right)$

265. $\rho = e^x$ for $x \in (0, 2)$

266. $\rho = x^3 + xe^{-x}$ for $x \in (0, 1)$

267. $\rho = x \sin x$ for $x \in (0, \pi)$

268. $\rho = \sqrt{x}$ for $x \in (1, 4)$

269. $\rho = \ln x$ for $x \in (1, e)$

For the following exercises, compute the center of mass $(\bar{x}, \bar{y})$. Use symmetry to help locate the center of mass whenever possible.

270. $\rho = 7$ in the square $0 \le x \le 1, \quad 0 \le y \le 1$

271. $\rho = 3$ in the triangle with vertices $(0, 0), \quad (a, 0)$, and $(0, b)$

272. $\rho = 2$ for the region bounded by $y = \cos(x)$, $y = -\cos(x), \quad x = -\frac{\pi}{2}$, and $x = \frac{\pi}{2}$

For the following exercises, use a calculator to draw the region, then compute the center of mass $(\bar{x}, \bar{y})$. Use symmetry to help locate the center of mass whenever possible.

273. **[T]** The region bounded by $y = \cos(2x)$, $x = -\frac{\pi}{4}$, and $x = \frac{\pi}{4}$

274. **[T]** The region between $y = 2x^2, \quad y = 0, \quad x = 0$, and $x = 1$

275. **[T]** The region between $y = \frac{5}{4}x^2$ and $y = 5$

276. **[T]** Region between $y = \sqrt{x}, \quad y = \ln(x), \quad x = 1$, and $x = 4$

277. **[T]** The region bounded by $y = 0, \quad \frac{x^2}{4} + \frac{y^2}{9} = 1$

278. **[T]** The region bounded by $y = 0, \quad x = 0$, and $\frac{x^2}{4} + \frac{y^2}{9} = 1$

279. **[T]** The region bounded by $y = x^2$ and $y = x^4$ in the first quadrant

For the following exercises, use the theorem of Pappus to determine the volume of the shape.

280. Rotating $y = mx$ around the x-axis between $x = 0$ and $x = 1$

281. Rotating $y = mx$ around the y-axis between $x = 0$ and $x = 1$

282. A general cone created by rotating a triangle with vertices $(0, 0), \quad (a, 0)$, and $(0, b)$ around the y-axis. Does your answer agree with the volume of a cone?

283. A general cylinder created by rotating a rectangle with vertices $(0, 0), \quad (a, 0), (0, b)$, and (a, b) around the y-axis. Does your answer agree with the volume of a cylinder?

284. A sphere created by rotating a semicircle with radius a around the y-axis. Does your answer agree with the volume of a sphere?

For the following exercises, use a calculator to draw the region enclosed by the curve. Find the area M and the centroid $(\overline{x}, \overline{y})$ for the given shapes. Use symmetry to help locate the center of mass whenever possible.

285. **[T]** Quarter-circle: $y = \sqrt{1 - x^2}$, $y = 0$, and $x = 0$

286. **[T]** Triangle: $y = x$, $y = 2 - x$, and $y = 0$

287. **[T]** Lens: $y = x^2$ and $y = x$

288. **[T]** Ring: $y^2 + x^2 = 1$ and $y^2 + x^2 = 4$

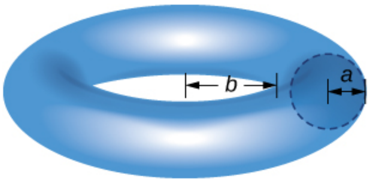
289. **[T]** Half-ring: $y^2 + x^2 = 1$, $y^2 + x^2 = 4$, and $y = 0$

290. Find the generalized center of mass in the sliver between $y = x^a$ and $y = x^b$ with $a > b$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the *y*-axis.

291. Find the generalized center of mass between $y = a^2 - x^2$, $x = 0$, and $y = 0$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the *y*-axis.

292. Find the generalized center of mass between $y = b\sin(ax)$, $x = 0$, and $x = \frac{\pi}{a}$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the *y*-axis.

293. Use the theorem of Pappus to find the volume of a torus (pictured here). Assume that a disk of radius a is positioned with the left end of the circle at $x = b$, $b > 0$, and is rotated around the *y*-axis.



294. Find the center of mass $(\overline{x}, \overline{y})$ for a thin wire along the semicircle $y = \sqrt{1 - x^2}$ with unit mass. (*Hint:* Use the theorem of Pappus.)

6.7 | Integrals, Exponential Functions, and Logarithms

Learning Objectives

- **6.7.1** Write the definition of the natural logarithm as an integral.
- **6.7.2** Recognize the derivative of the natural logarithm.
- **6.7.3** Integrate functions involving the natural logarithmic function.
- **6.7.4** Define the number e through an integral.
- **6.7.5** Recognize the derivative and integral of the exponential function.
- **6.7.6** Prove properties of logarithms and exponential functions using integrals.
- **6.7.7** Express general logarithmic and exponential functions in terms of natural logarithms and exponentials.

We already examined exponential functions and logarithms in earlier chapters. However, we glossed over some key details in the previous discussions. For example, we did not study how to treat exponential functions with exponents that are irrational. The definition of the number e is another area where the previous development was somewhat incomplete. We now have the tools to deal with these concepts in a more mathematically rigorous way, and we do so in this section.

For purposes of this section, assume we have not yet defined the natural logarithm, the number e, or any of the integration and differentiation formulas associated with these functions. By the end of the section, we will have studied these concepts in a mathematically rigorous way (and we will see they are consistent with the concepts we learned earlier).

We begin the section by defining the natural logarithm in terms of an integral. This definition forms the foundation for the section. From this definition, we derive differentiation formulas, define the number e, and expand these concepts to logarithms and exponential functions of any base.

The Natural Logarithm as an Integral

Recall the power rule for integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ \ n \neq -1.$$

Clearly, this does not work when $n = -1$, as it would force us to divide by zero. So, what do we do with $\int \frac{1}{x}dx$? Recall from the Fundamental Theorem of Calculus that $\int_1^x \frac{1}{t}dt$ is an antiderivative of $1/x$. Therefore, we can make the following definition.

Definition

For $x > 0$, define the natural logarithm function by

$$\ln x = \int_1^x \frac{1}{t}dt. \tag{6.24}$$

For $x > 1$, this is just the area under the curve $y = 1/t$ from 1 to x. For $x < 1$, we have $\int_1^x \frac{1}{t}dt = -\int_x^1 \frac{1}{t}dt$, so in this case it is the negative of the area under the curve from x to 1 (see the following figure).

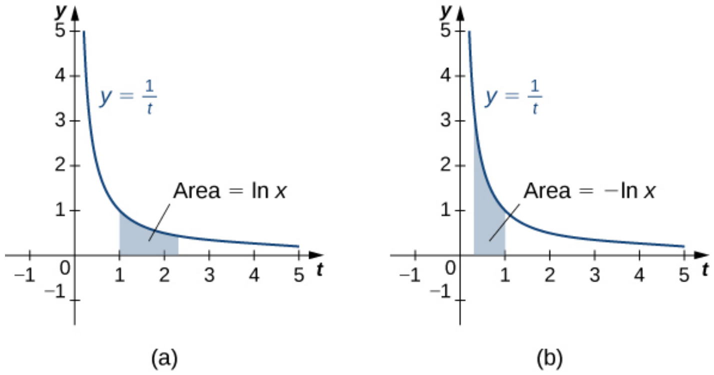


Figure 6.75 (a) When $x > 1,$ the natural logarithm is the area under the curve $y = 1/t$ from 1 to x. (b) When $x < 1,$ the natural logarithm is the negative of the area under the curve from x to 1.

Notice that $\ln 1 = 0$. Furthermore, the function $y = 1/t > 0$ for $x > 0$. Therefore, by the properties of integrals, it is clear that $\ln x$ is increasing for $x > 0$.

Properties of the Natural Logarithm

Because of the way we defined the natural logarithm, the following differentiation formula falls out immediately as a result of to the Fundamental Theorem of Calculus.

Theorem 6.15: Derivative of the Natural Logarithm

For $x > 0,$ the derivative of the natural logarithm is given by

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

Theorem 6.16: Corollary to the Derivative of the Natural Logarithm

The function $\ln x$ is differentiable; therefore, it is continuous.

A graph of $\ln x$ is shown in **Figure 6.76**. Notice that it is continuous throughout its domain of $(0, \infty)$.

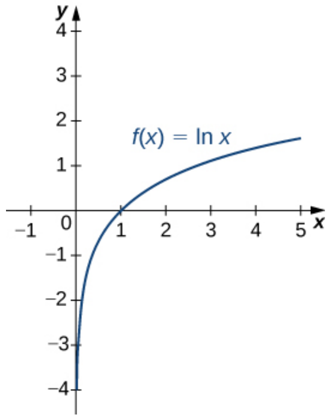


Figure 6.76 The graph of $f(x) = \ln x$ shows that it is a continuous function.

Example 6.35

Calculating Derivatives of Natural Logarithms

Calculate the following derivatives:

a. $\frac{d}{dx}\ln\left(5x^3 - 2\right)$

b. $\frac{d}{dx}(\ln(3x))^2$

Solution

We need to apply the chain rule in both cases.

a. $\frac{d}{dx}\ln\left(5x^3 - 2\right) = \frac{15x^2}{5x^3 - 2}$

b. $\frac{d}{dx}(\ln(3x))^2 = \frac{2(\ln(3x)) \cdot 3}{3x} = \frac{2(\ln(3x))}{x}$

6.35 Calculate the following derivatives:

a. $\frac{d}{dx}\ln\left(2x^2 + x\right)$

b. $\frac{d}{dx}\left(\ln\left(x^3\right)\right)^2$

Note that if we use the absolute value function and create a new function $\ln|x|$, we can extend the domain of the natural logarithm to include $x < 0$. Then $(d/(dx))\ln|x| = 1/x$. This gives rise to the familiar integration formula.

Theorem 6.17: Integral of (1/*u*) *du*

The natural logarithm is the antiderivative of the function $f(u) = 1/u$:

$$\int \frac{1}{u}du = \ln|u| + C.$$

Example 6.36

Calculating Integrals Involving Natural Logarithms

Calculate the integral $\int \frac{x}{x^2+4}dx.$

Solution

Using u-substitution, let $u = x^2 + 4$. Then $du = 2x\,dx$ and we have

$$\int \frac{x}{x^2+4}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2+4\right| + C = \frac{1}{2}\ln\left(x^2+4\right) + C.$$

6.36 Calculate the integral $\int \frac{x^2}{x^3+6}dx.$

Although we have called our function a “logarithm,” we have not actually proved that any of the properties of logarithms hold for this function. We do so here.

Theorem 6.18: Properties of the Natural Logarithm

If $a, b > 0$ and r is a rational number, then

i. $\ln 1 = 0$

ii. $\ln(ab) = \ln a + \ln b$

iii. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

iv. $\ln(a^r) = r\ln a$

Proof

i. By definition, $\ln 1 = \int_1^1 \frac{1}{t}dt = 0.$

ii. We have

$$() \ln(ab) = \int_1^{ab} \frac{1}{t}dt = \int_1^a \frac{1}{t}dt + \int_a^{ab} \frac{1}{t}dt.$$

Use u-substitution on the last integral in this expression. Let $u = t/a$. Then $du = (1/a)dt$. Furthermore, when $t = a,\ u = 1,$ and when $t = ab,\ u = b.$ So we get

$$() \ln(ab) = \int_1^a \frac{1}{t}dt + \int_a^{ab} \frac{1}{t}dt = \int_1^a \frac{1}{t}dt + \int_a^{ab} \frac{a}{t}\cdot\frac{1}{a}dt = \int_1^a \frac{1}{t}dt + \int_1^b \frac{1}{u}du = \ln a + \ln b.$$

iv. Note that

() $$\frac{d}{dx}\ln(x^r) = \frac{rx^{r-1}}{x^r} = \frac{r}{x}.$$

Furthermore,

() $$\frac{d}{dx}(r \ln x) = \frac{r}{x}.$$

Since the derivatives of these two functions are the same, by the Fundamental Theorem of Calculus, they must differ by a constant. So we have

() $$\ln(x^r) = r \ln x + C$$

for some constant C. Taking $x = 1,$ we get

() $$\begin{aligned} \ln(1^r) &= r\ln(1) + C \\ 0 &= r(0) + C \\ C &= 0. \end{aligned}$$

Thus $\ln(x^r) = r \ln x$ and the proof is complete. Note that we can extend this property to irrational values of r later in this section.
Part iii. follows from parts ii. and iv. and the proof is left to you.

□

Example 6.37

Using Properties of Logarithms

Use properties of logarithms to simplify the following expression into a single logarithm:

$$\ln 9 - 2\ln 3 + \ln\left(\frac{1}{3}\right).$$

Solution

We have

$$\ln 9 - 2\ln 3 + \ln\left(\frac{1}{3}\right) = \ln\left(3^2\right) - 2\ln 3 + \ln\left(3^{-1}\right) = 2\ln 3 - 2\ln 3 - \ln 3 = -\ln 3.$$

6.37 Use properties of logarithms to simplify the following expression into a single logarithm:

$$\ln 8 - \ln 2 - \ln\left(\frac{1}{4}\right).$$

Defining the Number *e*

Now that we have the natural logarithm defined, we can use that function to define the number e.

Definition

The number e is defined to be the real number such that

$$\ln e = 1.$$

To put it another way, the area under the curve $y = 1/t$ between $t = 1$ and $t = e$ is 1 (**Figure 6.77**). The proof that such a number exists and is unique is left to you. (*Hint*: Use the Intermediate Value Theorem to prove existence and the fact that $\ln x$ is increasing to prove uniqueness.)

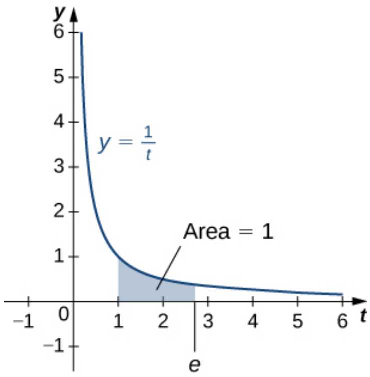


Figure 6.77 The area under the curve from 1 to e is equal to one.

The number e can be shown to be irrational, although we won't do so here (see the Student Project in **Taylor and Maclaurin Series (http://cnx.org/content/m53817/latest/)**). Its approximate value is given by

$$e \approx 2.71828182846.$$

The Exponential Function

We now turn our attention to the function e^x. Note that the natural logarithm is one-to-one and therefore has an inverse function. For now, we denote this inverse function by $\exp x$. Then,

$$\exp(\ln x) = x \text{ for } x > 0 \text{ and } \ln(\exp x) = x \text{ for all } x.$$

The following figure shows the graphs of $\exp x$ and $\ln x$.

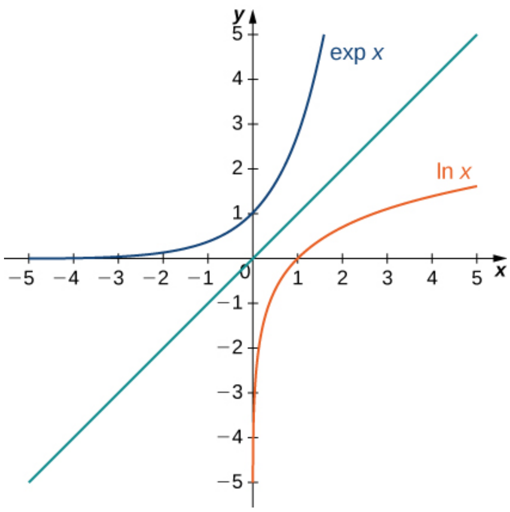


Figure 6.78 The graphs of $\ln x$ and $\exp x$.

We hypothesize that $\exp x = e^x$. For rational values of x, this is easy to show. If x is rational, then we have $\ln(e^x) = x \ln e = x$. Thus, when x is rational, $e^x = \exp x$. For irrational values of x, we simply define e^x as the inverse function of $\ln x$.

Definition

For any real number x, define $y = e^x$ to be the number for which

$$\ln y = \ln(e^x) = x. \tag{6.25}$$

Then we have $e^x = \exp(x)$ for all x, and thus

$$e^{\ln x} = x \text{ for } x > 0 \text{ and } \ln(e^x) = x \tag{6.26}$$

for all x.

Properties of the Exponential Function

Since the exponential function was defined in terms of an inverse function, and not in terms of a power of e, we must verify that the usual laws of exponents hold for the function e^x.

Theorem 6.19: Properties of the Exponential Function

If p and q are any real numbers and r is a rational number, then

i. $e^p e^q = e^{p+q}$

ii. $\dfrac{e^p}{e^q} = e^{p-q}$

iii. $(e^p)^r = e^{pr}$

Proof

Note that if p and q are rational, the properties hold. However, if p or q are irrational, we must apply the inverse function definition of e^x and verify the properties. Only the first property is verified here; the other two are left to you. We have

$$\ln(e^p e^q) = \ln(e^p) + \ln(e^q) = p + q = \ln\left(e^{p+q}\right).$$

Since $\ln x$ is one-to-one, then

$$e^p e^q = e^{p+q}.$$

□

As with part iv. of the logarithm properties, we can extend property iii. to irrational values of r, and we do so by the end of the section.

We also want to verify the differentiation formula for the function $y = e^x$. To do this, we need to use implicit differentiation. Let $y = e^x$. Then

$$\begin{aligned} \ln y &= x \\ \frac{d}{dx}\ln y &= \frac{d}{dx}x \\ \frac{1}{y}\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= y. \end{aligned}$$

Thus, we see

$$\frac{d}{dx}e^x = e^x$$

as desired, which leads immediately to the integration formula

$$\int e^x\,dx = e^x + C.$$

We apply these formulas in the following examples.

Example 6.38

Using Properties of Exponential Functions

Evaluate the following derivatives:

a. $\frac{d}{dt}e^{3t}e^{t^2}$

b. $\frac{d}{dx}e^{3x^2}$

Solution

We apply the chain rule as necessary.

a. $\frac{d}{dt}e^{3t}e^{t^2} = \frac{d}{dt}e^{3t+t^2} = e^{3t+t^2}(3+2t)$

b. $\frac{d}{dx}e^{3x^2} = e^{3x^2}6x$

6.38 Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\frac{e^{x^2}}{e^{5x}}\right)$

b. $\frac{d}{dt}\left(e^{2t}\right)^3$

Example 6.39

Using Properties of Exponential Functions

Evaluate the following integral: $\int 2xe^{-x^2}\,dx.$

Solution

Using u -substitution, let $u = -x^2$. Then $du = -2x\,dx$, and we have

$$\int 2xe^{-x^2}\,dx = -\int e^u\,du = -e^u + C = -e^{-x^2} + C.$$

6.39 Evaluate the following integral: $\int \frac{4}{e^{3x}} dx.$

General Logarithmic and Exponential Functions

We close this section by looking at exponential functions and logarithms with bases other than e. Exponential functions are functions of the form $f(x) = a^x$. Note that unless $a = e$, we still do not have a mathematically rigorous definition of these functions for irrational exponents. Let's rectify that here by defining the function $f(x) = a^x$ in terms of the exponential function e^x. We then examine logarithms with bases other than e as inverse functions of exponential functions.

Definition

For any $a > 0$, and for any real number x, define $y = a^x$ as follows:

$$y = a^x = e^{x \ln a}.$$

Now a^x is defined rigorously for all values of x. This definition also allows us to generalize property iv. of logarithms and property iii. of exponential functions to apply to both rational and irrational values of r. It is straightforward to show that properties of exponents hold for general exponential functions defined in this way.

Let's now apply this definition to calculate a differentiation formula for a^x. We have

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a} \ln a = a^x \ln a.$$

The corresponding integration formula follows immediately.

Theorem 6.20: Derivatives and Integrals Involving General Exponential Functions

Let $a > 0$. Then,

$$\frac{d}{dx}a^x = a^x \ln a$$

and

$$\int a^x \, dx = \frac{1}{\ln a}a^x + C.$$

If $a \neq 1$, then the function a^x is one-to-one and has a well-defined inverse. Its inverse is denoted by $\log_a x$. Then,

$$y = \log_a x \text{ if and only if } x = a^y.$$

Note that general logarithm functions can be written in terms of the natural logarithm. Let $y = \log_a x$. Then, $x = a^y$. Taking the natural logarithm of both sides of this second equation, we get

$$\begin{aligned} \ln x &= \ln(a^y) \\ \ln x &= y \ln a \\ y &= \frac{\ln x}{\ln a} \\ \log x &= \frac{\ln x}{\ln a}. \end{aligned}$$

Thus, we see that all logarithmic functions are constant multiples of one another. Next, we use this formula to find a differentiation formula for a logarithm with base a. Again, let $y = \log_a x$. Then,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\log_a x) \\ &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) \\ &= \left(\frac{1}{\ln a}\right)\frac{d}{dx}(\ln x) \\ &= \frac{1}{\ln a}\cdot\frac{1}{x} \\ &= \frac{1}{x \ln a}.\end{aligned}$$

Theorem 6.21: Derivatives of General Logarithm Functions

Let $a > 0$. Then,

$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}.$$

Example 6.40

Calculating Derivatives of General Exponential and Logarithm Functions

Evaluate the following derivatives:

a. $\frac{d}{dt}\left(4^t \cdot 2^{t^2}\right)$

b. $\frac{d}{dx}\log_8\left(7x^2 + 4\right)$

Solution

We need to apply the chain rule as necessary.

a. $\frac{d}{dt}\left(4^t \cdot 2^{t^2}\right) = \frac{d}{dt}\left(2^{2t} \cdot 2^{t^2}\right) = \frac{d}{dt}\left(2^{2t + t^2}\right) = 2^{2t + t^2}\ln(2)(2 + 2t)$

b. $\frac{d}{dx}\log_8\left(7x^2 + 4\right) = \frac{1}{\left(7x^2 + 4\right)(\ln 8)}(14x)$

6.40 Evaluate the following derivatives:

a. $\frac{d}{dt}4^{t^4}$

b. $\frac{d}{dx}\log_3\left(\sqrt{x^2 + 1}\right)$

Example 6.41

Integrating General Exponential Functions

Evaluate the following integral: $\int \frac{3}{2^{3x}} dx$.

Solution

Use u-substitution and let $u = -3x$. Then $du = -3dx$ and we have

$$\int \frac{3}{2^{3x}} dx = \int 3 \cdot 2^{-3x} dx = -\int 2^u \, du = -\frac{1}{\ln 2} 2^u + C = -\frac{1}{\ln 2} 2^{-3x} + C.$$

6.41 Evaluate the following integral: $\int x^2 2^{x^3} \, dx$.

6.7 EXERCISES

For the following exercises, find the derivative $\frac{dy}{dx}$.

295. $y = \ln(2x)$

296. $y = \ln(2x + 1)$

297. $y = \frac{1}{\ln x}$

For the following exercises, find the indefinite integral.

298. $\int \frac{dt}{3t}$

299. $\int \frac{dx}{1 + x}$

For the following exercises, find the derivative dy/dx. (You can use a calculator to plot the function and the derivative to confirm that it is correct.)

300. **[T]** $y = \frac{\ln(x)}{x}$

301. **[T]** $y = x \ln(x)$

302. **[T]** $y = \log_{10} x$

303. **[T]** $y = \ln(\sin x)$

304. **[T]** $y = \ln(\ln x)$

305. **[T]** $y = 7 \ln(4x)$

306. **[T]** $y = \ln\left((4x)^7\right)$

307. **[T]** $y = \ln(\tan x)$

308. **[T]** $y = \ln(\tan(3x))$

309. **[T]** $y = \ln\left(\cos^2 x\right)$

For the following exercises, find the definite or indefinite integral.

310. $\int_0^1 \frac{dx}{3 + x}$

311. $\int_0^1 \frac{dt}{3 + 2t}$

312. $\int_0^2 \frac{x\,dx}{x^2 + 1}$

313. $\int_0^2 \frac{x^3\,dx}{x^2 + 1}$

314. $\int_2^e \frac{dx}{x \ln x}$

315. $\int_2^e \frac{dx}{x(\ln x)^2}$

316. $\int \frac{\cos x\,dx}{\sin x}$

317. $\int_0^{\pi/4} \tan x\,dx$

318. $\int \cot(3x)dx$

319. $\int \frac{(\ln x)^2\,dx}{x}$

For the following exercises, compute dy/dx by differentiating $\ln y$.

320. $y = \sqrt{x^2 + 1}$

321. $y = \sqrt{x^2 + 1}\sqrt{x^2 - 1}$

322. $y = e^{\sin x}$

323. $y = x^{-1/x}$

324. $y = e^{(ex)}$

325. $y = x^e$

326. $y = x^{(ex)}$

327. $y = \sqrt{x}\sqrt[3]{x}\sqrt[6]{x}$

328. $y = x^{-1/\ln x}$

329. $y = e^{-\ln x}$

For the following exercises, evaluate by any method.

330. $\int_5^{10} \frac{dt}{t} - \int_{5x}^{10x} \frac{dt}{t}$

331. $\int_1^{e^\pi} \frac{dx}{x} + \int_{-2}^{-1} \frac{dx}{x}$

332. $\frac{d}{dx}\int_x^1 \frac{dt}{t}$

333. $\frac{d}{dx}\int_x^{x^2} \frac{dt}{t}$

334. $\frac{d}{dx}\ln(\sec x + \tan x)$

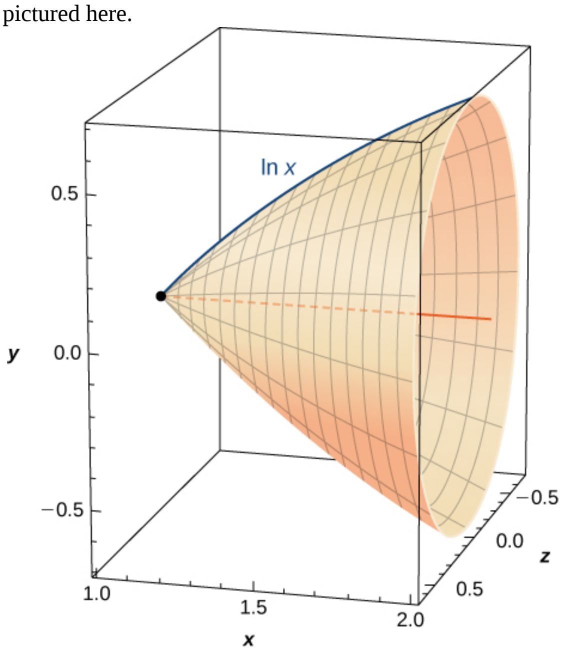
For the following exercises, use the function $\ln x$. If you are unable to find intersection points analytically, use a calculator.

335. Find the area of the region enclosed by $x = 1$ and $y = 5$ above $y = \ln x$.

336. **[T]** Find the arc length of $\ln x$ from $x = 1$ to $x = 2$.

337. Find the area between $\ln x$ and the x-axis from $x = 1$ to $x = 2$.

338. Find the volume of the shape created when rotating this curve from $x = 1$ to $x = 2$ around the x-axis, as pictured here.



339. **[T]** Find the surface area of the shape created when rotating the curve in the previous exercise from $x = 1$ to $x = 2$ around the x-axis.

If you are unable to find intersection points analytically in the following exercises, use a calculator.

340. Find the area of the hyperbolic quarter-circle enclosed by $x = 2$ and $y = 2$ above $y = 1/x$.

341. **[T]** Find the arc length of $y = 1/x$ from $x = 1$ to $x = 4$.

342. Find the area under $y = 1/x$ and above the x-axis from $x = 1$ to $x = 4$.

For the following exercises, verify the derivatives and antiderivatives.

343. $\frac{d}{dx}\ln\left(x + \sqrt{x^2 + 1}\right) = \frac{1}{\sqrt{1 + x^2}}$

344. $\frac{d}{dx}\ln\left(\frac{x - a}{x + a}\right) = \frac{2a}{\left(x^2 - a^2\right)}$

345. $\frac{d}{dx}\ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right) = -\frac{1}{x\sqrt{1 - x^2}}$

346. $\frac{d}{dx}\ln\left(x + \sqrt{x^2 - a^2}\right) = \frac{1}{\sqrt{x^2 - a^2}}$

347. $\int \frac{dx}{x\ln(x)\ln(\ln x)} = \ln(\ln(\ln x)) + C$

6.8 | Exponential Growth and Decay

Learning Objectives

6.8.1 Use the exponential growth model in applications, including population growth and compound interest.
6.8.2 Explain the concept of doubling time.
6.8.3 Use the exponential decay model in applications, including radioactive decay and Newton's law of cooling.
6.8.4 Explain the concept of half-life.

One of the most prevalent applications of exponential functions involves growth and decay models. Exponential growth and decay show up in a host of natural applications. From population growth and continuously compounded interest to radioactive decay and Newton's law of cooling, exponential functions are ubiquitous in nature. In this section, we examine exponential growth and decay in the context of some of these applications.

Exponential Growth Model

Many systems exhibit exponential growth. These systems follow a model of the form $y = y_0 e^{kt}$, where y_0 represents the initial state of the system and k is a positive constant, called the *growth constant*. Notice that in an exponential growth model, we have

$$y' = ky_0 e^{kt} = ky. \tag{6.27}$$

That is, the rate of growth is proportional to the current function value. This is a key feature of exponential growth. **Equation 6.27** involves derivatives and is called a *differential equation*. We learn more about differential equations in **Introduction to Differential Equations (http://cnx.org/content/m53696/latest/)** .

Rule: Exponential Growth Model

Systems that exhibit **exponential growth** increase according to the mathematical model

$$y = y_0 e^{kt},$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the *growth constant*.

Population growth is a common example of exponential growth. Consider a population of bacteria, for instance. It seems plausible that the rate of population growth would be proportional to the size of the population. After all, the more bacteria there are to reproduce, the faster the population grows. **Figure 6.79** and **Table 6.1** represent the growth of a population of bacteria with an initial population of 200 bacteria and a growth constant of $0.02.$ Notice that after only 2 hours $(120$ minutes), the population is 10 times its original size!

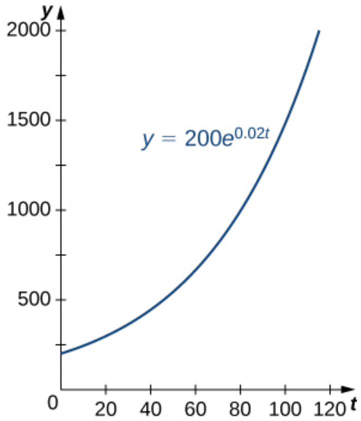


Figure 6.79 An example of exponential growth for bacteria.

Time (min)	Population Size (no. of bacteria)
10	244
20	298
30	364
40	445
50	544
60	664
70	811
80	991
90	1210
100	1478
110	1805
120	2205

Table 6.1 Exponential Growth of a Bacterial Population

Note that we are using a continuous function to model what is inherently discrete behavior. At any given time, the real-world population contains a whole number of bacteria, although the model takes on noninteger values. When using exponential growth models, we must always be careful to interpret the function values in the context of the phenomenon we are modeling.

Example 6.42

Population Growth

Consider the population of bacteria described earlier. This population grows according to the function $f(t) = 200e^{0.02t}$, where t is measured in minutes. How many bacteria are present in the population after 5 hours $(300$ minutes)? When does the population reach $100,000$ bacteria?

Solution

We have $f(t) = 200e^{0.02t}$. Then

$$f(300) = 200e^{0.02(300)} \approx 80,686.$$

There are $80,686$ bacteria in the population after 5 hours.

To find when the population reaches $100,000$ bacteria, we solve the equation

$$\begin{aligned} 100,000 &= 200e^{0.02t} \\ 500 &= e^{0.02t} \\ \ln 500 &= 0.02t \\ t &= \frac{\ln 500}{0.02} \approx 310.73. \end{aligned}$$

The population reaches $100,000$ bacteria after 310.73 minutes.

6.42 Consider a population of bacteria that grows according to the function $f(t) = 500e^{0.05t}$, where t is measured in minutes. How many bacteria are present in the population after 4 hours? When does the population reach 100 million bacteria?

Let's now turn our attention to a financial application: compound interest. Interest that is not compounded is called *simple interest*. Simple interest is paid once, at the end of the specified time period (usually 1 year). So, if we put $\$1000$ in a savings account earning 2% simple interest per year, then at the end of the year we have

$$1000(1 + 0.02) = \$1020.$$

Compound interest is paid multiple times per year, depending on the compounding period. Therefore, if the bank compounds the interest every 6 months, it credits half of the year's interest to the account after 6 months. During the second half of the year, the account earns interest not only on the initial $\$1000$, but also on the interest earned during the first half of the year. Mathematically speaking, at the end of the year, we have

$$1000\left(1 + \frac{0.02}{2}\right)^2 = \$1020.10.$$

Similarly, if the interest is compounded every 4 months, we have

$$1000\left(1 + \frac{0.02}{3}\right)^3 = \$1020.13,$$

and if the interest is compounded daily $(365$ times per year), we have $\$1020.20.$ If we extend this concept, so that the interest is compounded continuously, after t years we have

$$1000 \lim_{n \to \infty} \left(1 + \frac{0.02}{n}\right)^{nt}.$$

Now let's manipulate this expression so that we have an exponential growth function. Recall that the number e can be expressed as a limit:

$$e = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{m}.$$

Based on this, we want the expression inside the parentheses to have the form $(1 + 1/m)$. Let $n = 0.02m$. Note that as $n \to \infty$, $m \to \infty$ as well. Then we get

$$1000 \lim_{n \to \infty} \left(1 + \frac{0.02}{n}\right)^{nt} = 1000 \lim_{m \to \infty} \left(1 + \frac{0.02}{0.02m}\right)^{0.02mt} = 1000\left[\lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{m}\right]^{0.02t}.$$

We recognize the limit inside the brackets as the number e. So, the balance in our bank account after t years is given by $1000e^{0.02t}$. Generalizing this concept, we see that if a bank account with an initial balance of $\$P$ earns interest at a rate of $r\%$, compounded continuously, then the balance of the account after t years is

$$\text{Balance} = Pe^{rt}.$$

Example 6.43

Compound Interest

A 25-year-old student is offered an opportunity to invest some money in a retirement account that pays 5% annual interest compounded continuously. How much does the student need to invest today to have $\$1$ million when she retires at age 65? What if she could earn 6% annual interest compounded continuously instead?

Solution

We have

$$\begin{aligned} 1{,}000{,}000 &= Pe^{0.05(40)} \\ P &= 135{,}335.28. \end{aligned}$$

She must invest $\$135{,}335.28$ at 5% interest.

If, instead, she is able to earn 6%, then the equation becomes

$$\begin{aligned} 1{,}000{,}000 &= Pe^{0.06(40)} \\ P &= 90{,}717.95. \end{aligned}$$

In this case, she needs to invest only $\$90{,}717.95$. This is roughly two-thirds the amount she needs to invest at 5%. The fact that the interest is compounded continuously greatly magnifies the effect of the 1% increase in interest rate.

6.43 Suppose instead of investing at age 25, the student waits until age 35. How much would she have to invest at 5%? At 6%?

If a quantity grows exponentially, the time it takes for the quantity to double remains constant. In other words, it takes the same amount of time for a population of bacteria to grow from 100 to 200 bacteria as it does to grow from $10{,}000$ to $20{,}000$ bacteria. This time is called the doubling time. To calculate the doubling time, we want to know when the quantity reaches twice its original size. So we have

$$\begin{aligned} 2y_0 &= y_0 e^{kt} \\ 2 &= e^{kt} \\ \ln 2 &= kt \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

Definition

If a quantity grows exponentially, the **doubling time** is the amount of time it takes the quantity to double. It is given by

$$\text{Doubling time } = \frac{\ln 2}{k}.$$

Example 6.44

Using the Doubling Time

Assume a population of fish grows exponentially. A pond is stocked initially with 500 fish. After 6 months, there are 1000 fish in the pond. The owner will allow his friends and neighbors to fish on his pond after the fish population reaches $10{,}000.$ When will the owner's friends be allowed to fish?

Solution

We know it takes the population of fish 6 months to double in size. So, if t represents time in months, by the doubling-time formula, we have $6 = (\ln 2)/k.$ Then, $k = (\ln 2)/6.$ Thus, the population is given by $y = 500e^{((\ln 2)/6)t}.$ To figure out when the population reaches $10{,}000$ fish, we must solve the following equation:

$$\begin{aligned} 10{,}000 &= 500e^{(\ln 2/6)t} \\ 20 &= e^{(\ln 2/6)t} \\ \ln 20 &= \left(\frac{\ln 2}{6}\right)t \\ t &= \frac{6(\ln 20)}{\ln 2} \approx 25.93. \end{aligned}$$

The owner's friends have to wait 25.93 months (a little more than 2 years) to fish in the pond.

6.44 Suppose it takes 9 months for the fish population in **Example 6.44** to reach 1000 fish. Under these circumstances, how long do the owner's friends have to wait?

Exponential Decay Model

Exponential functions can also be used to model populations that shrink (from disease, for example), or chemical compounds that break down over time. We say that such systems exhibit exponential decay, rather than exponential growth. The model is nearly the same, except there is a negative sign in the exponent. Thus, for some positive constant k, we have $y = y_0 e^{-kt}.$

As with exponential growth, there is a differential equation associated with exponential decay. We have

$$y' = -ky_0 e^{-kt} = -ky.$$

Rule: Exponential Decay Model

Systems that exhibit **exponential decay** behave according to the model

$$y = y_0 e^{-kt},$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the *decay constant*.

The following figure shows a graph of a representative exponential decay function.

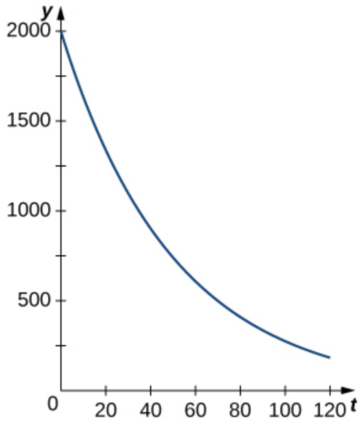


Figure 6.80 An example of exponential decay.

Let’s look at a physical application of exponential decay. Newton’s law of cooling says that an object cools at a rate proportional to the difference between the temperature of the object and the temperature of the surroundings. In other words, if T represents the temperature of the object and T_a represents the ambient temperature in a room, then

$$T' = -k(T - T_a).$$

Note that this is not quite the right model for exponential decay. We want the derivative to be proportional to the function, and this expression has the additional T_a term. Fortunately, we can make a change of variables that resolves this issue. Let $y(t) = T(t) - T_a$. Then $y'(t) = T'(t) - 0 = T'(t)$, and our equation becomes

$$y' = -ky.$$

From our previous work, we know this relationship between *y* and its derivative leads to exponential decay. Thus,

$$y = y_0 e^{-kt},$$

and we see that

$$\begin{aligned} T - T_a &= (T_0 - T_a)e^{-kt} \\ T &= (T_0 - T_a)e^{-kt} + T_a \end{aligned}$$

where T_0 represents the initial temperature. Let’s apply this formula in the following example.

Example 6.45

Newton’s Law of Cooling

According to experienced baristas, the optimal temperature to serve coffee is between $155°\text{F}$ and $175°\text{F}$. Suppose coffee is poured at a temperature of $200°\text{F}$, and after 2 minutes in a $70°\text{F}$ room it has cooled to

$180°F.$ When is the coffee first cool enough to serve? When is the coffee too cold to serve? Round answers to the nearest half minute.

Solution

We have

$$\begin{aligned}
T &= (T_0 - T_a)e^{-kt} + T_a \\
180 &= (200 - 70)e^{-k(2)} + 70 \\
110 &= 130e^{-2k} \\
\frac{11}{13} &= e^{-2k} \\
\ln \frac{11}{13} &= -2k \\
\ln 11 - \ln 13 &= -2k \\
k &= \frac{\ln 13 - \ln 11}{2}.
\end{aligned}$$

Then, the model is

$$T = 130e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} + 70.$$

The coffee reaches $175°F$ when

$$\begin{aligned}
175 &= 130e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} + 70 \\
105 &= 130e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} \\
\frac{21}{26} &= e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} \\
\ln \frac{21}{26} &= \frac{\ln 11 - \ln 13}{2}t \\
\ln 21 - \ln 26 &= \frac{\ln 11 - \ln 13}{2}t \\
t &= \frac{2(\ln 21 - \ln 26)}{\ln 11 - \ln 13} \approx 2.56.
\end{aligned}$$

The coffee can be served about 2.5 minutes after it is poured. The coffee reaches $155°F$ at

$$\begin{aligned}
155 &= 130e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} + 70 \\
85 &= 130e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} \\
\frac{17}{26} &= e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} \\
\ln 17 - \ln 26 &= \left(\frac{\ln 11 - \ln 13}{2}\right)t \\
t &= \frac{2(\ln 17 - \ln 26)}{\ln 11 - \ln 13} \approx 5.09.
\end{aligned}$$

The coffee is too cold to be served about 5 minutes after it is poured.

6.45 Suppose the room is warmer $(75°F)$ and, after 2 minutes, the coffee has cooled only to $185°F.$ When is the coffee first cool enough to serve? When is the coffee be too cold to serve? Round answers to the nearest half minute.

Just as systems exhibiting exponential growth have a constant doubling time, systems exhibiting exponential decay have a constant half-life. To calculate the half-life, we want to know when the quantity reaches half its original size. Therefore, we have

$$\begin{aligned} \frac{y_0}{2} &= y_0 e^{-kt} \\ \frac{1}{2} &= e^{-kt} \\ -\ln 2 &= -kt \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

Note: This is the same expression we came up with for doubling time.

Definition

If a quantity decays exponentially, the **half-life** is the amount of time it takes the quantity to be reduced by half. It is given by

$$\text{Half-life} = \frac{\ln 2}{k}.$$

Example 6.46

Radiocarbon Dating

One of the most common applications of an exponential decay model is carbon dating. Carbon-14 decays (emits a radioactive particle) at a regular and consistent exponential rate. Therefore, if we know how much carbon was originally present in an object and how much carbon remains, we can determine the age of the object. The half-life of carbon-14 is approximately 5730 years—meaning, after that many years, half the material has converted from the original carbon-14 to the new nonradioactive nitrogen-14. If we have 100 g carbon-14 today, how much is left in 50 years? If an artifact that originally contained 100 g of carbon now contains 10 g of carbon, how old is it? Round the answer to the nearest hundred years.

Solution

We have

$$\begin{aligned} 5730 &= \frac{\ln 2}{k} \\ k &= \frac{\ln 2}{5730}. \end{aligned}$$

So, the model says

$$y = 100e^{-(\ln 2/5730)t}.$$

In 50 years, we have

$$\begin{aligned} y &= 100e^{-(\ln 2/5730)(50)} \\ &\approx 99.40. \end{aligned}$$

Therefore, in 50 years, 99.40 g of carbon-14 remains.

To determine the age of the artifact, we must solve

$$\begin{aligned} 10 &= 100e^{-(\ln 2/5730)t} \\ \frac{1}{10} &= e^{-(\ln 2/5730)t} \\ t &\approx 19035. \end{aligned}$$

The artifact is about $19{,}000$ years old.

6.46 If we have 100 g of carbon-14, how much is left after t years? If an artifact that originally contained 100 g of carbon now contains $20g$ of carbon, how old is it? Round the answer to the nearest hundred years.

6.8 EXERCISES

True or False? If true, prove it. If false, find the true answer.

348. The doubling time for $y = e^{ct}$ is $(\ln(2))/(\ln(c))$.

349. If you invest $\$500$, an annual rate of interest of 3% yields more money in the first year than a 2.5% continuous rate of interest.

350. If you leave a $100°\text{C}$ pot of tea at room temperature $(25°\text{C})$ and an identical pot in the refrigerator $(5°\text{C})$, with $k = 0.02$, the tea in the refrigerator reaches a drinkable temperature $(70°\text{C})$ more than 5 minutes before the tea at room temperature.

351. If given a half-life of t years, the constant k for $y = e^{kt}$ is calculated by $k = \ln(1/2)/t$.

For the following exercises, use $y = y_0 e^{kt}$.

352. If a culture of bacteria doubles in 3 hours, how many hours does it take to multiply by 10?

353. If bacteria increase by a factor of 10 in 10 hours, how many hours does it take to increase by 100?

354. How old is a skull that contains one-fifth as much radiocarbon as a modern skull? Note that the half-life of radiocarbon is 5730 years.

355. If a relic contains 90% as much radiocarbon as new material, can it have come from the time of Christ (approximately 2000 years ago)? Note that the half-life of radiocarbon is 5730 years.

356. The population of Cairo grew from 5 million to 10 million in 20 years. Use an exponential model to find when the population was 8 million.

357. The populations of New York and Los Angeles are growing at 1% and 1.4% a year, respectively. Starting from 8 million (New York) and 6 million (Los Angeles), when are the populations equal?

358. Suppose the value of $\$1$ in Japanese yen decreases at 2% per year. Starting from $\$1 = ¥250$, when will $\$1 = ¥1$?

359. The effect of advertising decays exponentially. If 40% of the population remembers a new product after 3 days, how long will 20% remember it?

360. If $y = 1000$ at $t = 3$ and $y = 3000$ at $t = 4$, what was y_0 at $t = 0$?

361. If $y = 100$ at $t = 4$ and $y = 10$ at $t = 8$, when does $y = 1$?

362. If a bank offers annual interest of 7.5% or continuous interest of 7.25%, which has a better annual yield?

363. What continuous interest rate has the same yield as an annual rate of 9%?

364. If you deposit $\$5000$ at 8% annual interest, how many years can you withdraw $\$500$ (starting after the first year) without running out of money?

365. You are trying to save $\$50,000$ in 20 years for college tuition for your child. If interest is a continuous 10%, how much do you need to invest initially?

366. You are cooling a turkey that was taken out of the oven with an internal temperature of $165°\text{F}$. After 10 minutes of resting the turkey in a $70°\text{F}$ apartment, the temperature has reached $155°\text{F}$. What is the temperature of the turkey 20 minutes after taking it out of the oven?

367. You are trying to thaw some vegetables that are at a temperature of $1°\text{F}$. To thaw vegetables safely, you must put them in the refrigerator, which has an ambient temperature of $44°\text{F}$. You check on your vegetables 2 hours after putting them in the refrigerator to find that they are now $12°\text{F}$. Plot the resulting temperature curve and use it to determine when the vegetables reach $33°\text{F}$.

368. You are an archaeologist and are given a bone that is claimed to be from a Tyrannosaurus Rex. You know these dinosaurs lived during the Cretaceous Era $(146$ million years to 65 million years ago), and you find by radiocarbon dating that there is 0.000001% the amount of radiocarbon. Is this bone from the Cretaceous?

369. The spent fuel of a nuclear reactor contains plutonium-239, which has a half-life of $24,000$ years. If 1 barrel containing $10\,\text{kg}$ of plutonium-239 is sealed, how many years must pass until only $10g$ of plutonium-239 is left?

For the next set of exercises, use the following table, which features the world population by decade.

Years since 1950	Population (millions)
0	2,556
10	3,039
20	3,706
30	4,453
40	5,279
50	6,083
60	6,849

Source: http://www.factmonster.com/ipka/A0762181.html.

370. **[T]** The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 2686e^{0.01604t}$. Use a graphing calculator to graph the data and the exponential curve together.

371. **[T]** Find and graph the derivative y' of your equation. Where is it increasing and what is the meaning of this increase?

372. **[T]** Find and graph the second derivative of your equation. Where is it increasing and what is the meaning of this increase?

373. **[T]** Find the predicted date when the population reaches 10 billion. Using your previous answers about the first and second derivatives, explain why exponential growth is unsuccessful in predicting the future.

For the next set of exercises, use the following table, which shows the population of San Francisco during the 19th century.

Years since 1850	Population (thousands)
0	21.00
10	56.80
20	149.5
30	234.0

Source: http://www.sfgenealogy.com/sf/history/hgpop.htm.

374. **[T]** The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 35.26e^{0.06407t}$. Use a graphing calculator to graph the data and the exponential curve together.

375. **[T]** Find and graph the derivative y' of your equation. Where is it increasing? What is the meaning of this increase? Is there a value where the increase is maximal?

376. **[T]** Find and graph the second derivative of your equation. Where is it increasing? What is the meaning of this increase?

6.9 | Calculus of the Hyperbolic Functions

Learning Objectives

6.9.1 Apply the formulas for derivatives and integrals of the hyperbolic functions.

6.9.2 Apply the formulas for the derivatives of the inverse hyperbolic functions and their associated integrals.

6.9.3 Describe the common applied conditions of a catenary curve.

We were introduced to hyperbolic functions in **Introduction to Functions and Graphs**, along with some of their basic properties. In this section, we look at differentiation and integration formulas for the hyperbolic functions and their inverses.

Derivatives and Integrals of the Hyperbolic Functions

Recall that the hyperbolic sine and hyperbolic cosine are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}.$$

The other hyperbolic functions are then defined in terms of $\sinh x$ and $\cosh x$. The graphs of the hyperbolic functions are shown in the following figure.

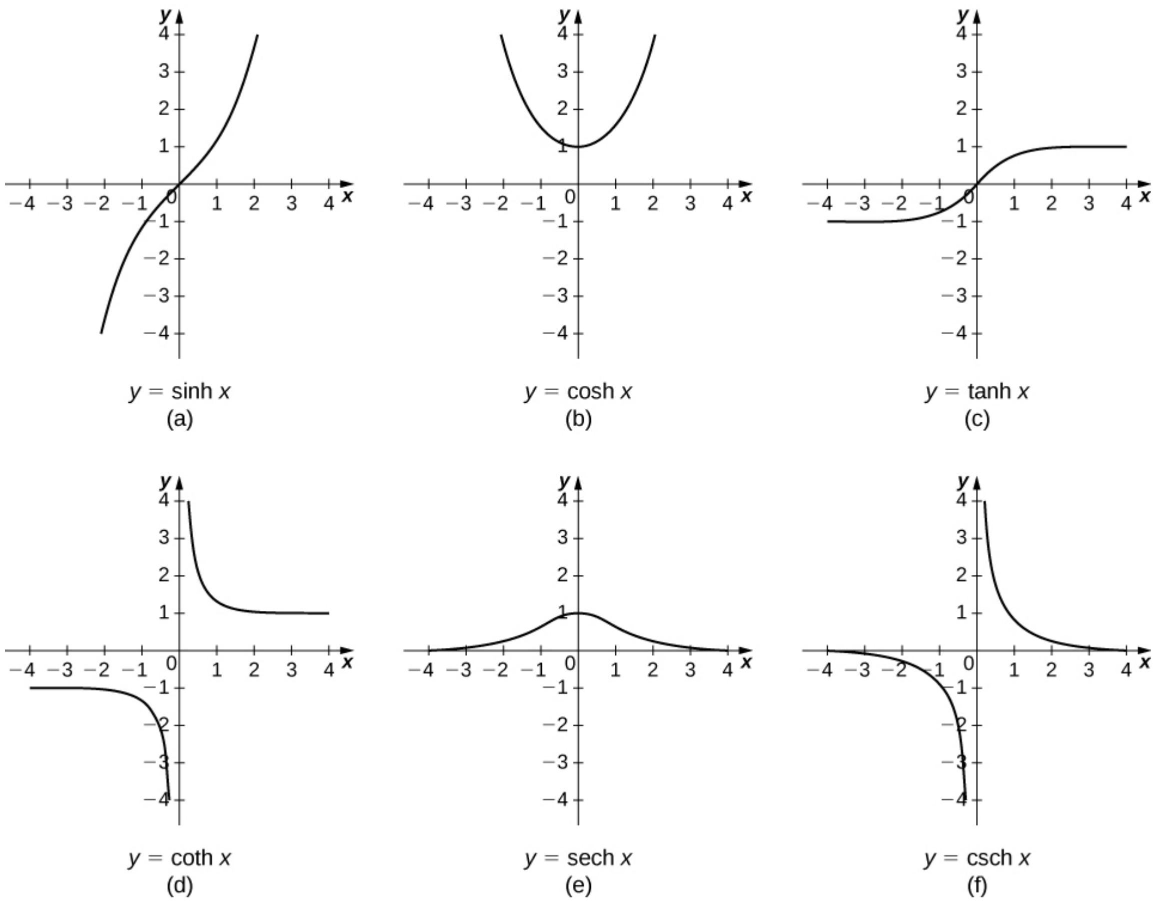


Figure 6.81 Graphs of the hyperbolic functions.

It is easy to develop differentiation formulas for the hyperbolic functions. For example, looking at $\sinh x$ we have

$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)\\ &= \frac{1}{2}\left[\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right]\\ &= \frac{1}{2}[e^x + e^{-x}] = \cosh x.\end{aligned}$$

Similarly, $(d/dx)\cosh x = \sinh x$. We summarize the differentiation formulas for the hyperbolic functions in the following table.

$f(x)$	$\frac{d}{dx}f(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2\, x$
$\coth x$	$-\text{csch}^2\, x$
$\text{sech}\, x$	$-\text{sech}\, x \tanh x$
$\text{csch}\, x$	$-\text{csch}\, x \coth x$

Table 6.2 Derivatives of the Hyperbolic Functions

Let's take a moment to compare the derivatives of the hyperbolic functions with the derivatives of the standard trigonometric functions. There are a lot of similarities, but differences as well. For example, the derivatives of the sine functions match: $(d/dx)\sin x = \cos x$ and $(d/dx)\sinh x = \cosh x$. The derivatives of the cosine functions, however, differ in sign: $(d/dx)\cos x = -\sin x$, but $(d/dx)\cosh x = \sinh x$. As we continue our examination of the hyperbolic functions, we must be mindful of their similarities and differences to the standard trigonometric functions.

These differentiation formulas for the hyperbolic functions lead directly to the following integral formulas.

$$\begin{aligned}\int \sinh u\,du &= \cosh u + C & \int \text{csch}^2\, u\,du &= -\coth u + C\\ \int \cosh u\,du &= \sinh u + C & \int \text{sech}\, u \tanh u\,du &= -\text{sech}\, u + C\\ \int \text{sech}^2 u\,du &= \tanh u + C & \int \text{csch}\, u \coth u\,du &= -\text{csch}\, u + C\end{aligned}$$

Example 6.47

Differentiating Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\sinh\left(x^2\right)\right)$

b. $\frac{d}{dx}(\cosh x)^2$

Solution

Using the formulas in **Table 6.2** and the chain rule, we get

a. $\frac{d}{dx}\left(\sinh\left(x^2\right)\right) = \cosh\left(x^2\right) \cdot 2x$

b. $\frac{d}{dx}(\cosh x)^2 = 2 \cosh x \sinh x$

6.47 Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\tanh\left(x^2 + 3x\right)\right)$

b. $\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right)$

Example 6.48

Integrals Involving Hyperbolic Functions

Evaluate the following integrals:

a. $\int x \cosh\left(x^2\right) dx$

b. $\int \tanh x \, dx$

Solution

We can use u-substitution in both cases.

a. Let $u = x^2$. Then, $du = 2x\,dx$ and

$$\int x \cosh\left(x^2\right) dx = \int \frac{1}{2} \cosh u \, du = \frac{1}{2} \sinh u + C = \frac{1}{2} \sinh\left(x^2\right) + C.$$

b. Let $u = \cosh x$. Then, $du = \sinh x \, dx$ and

$$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} dx = \int \frac{1}{u} du = \ln|u| + C = \ln|\cosh x| + C.$$

Note that $\cosh x > 0$ for all x, so we can eliminate the absolute value signs and obtain

$$\int \tanh x \, dx = \ln(\cosh x) + C.$$

6.48 Evaluate the following integrals:

a. $\int \sinh^3 x \cosh x \, dx$

b. $\int \text{sech}^2 (3x) dx$

Calculus of Inverse Hyperbolic Functions

Looking at the graphs of the hyperbolic functions, we see that with appropriate range restrictions, they all have inverses. Most of the necessary range restrictions can be discerned by close examination of the graphs. The domains and ranges of the inverse hyperbolic functions are summarized in the following table.

Function	Domain	Range
$\sinh^{-1} x$	$(-\infty, \infty)$	$(-\infty, \infty)$
$\cosh^{-1} x$	$[1, \infty)$	$[0, \infty)$
$\tanh^{-1} x$	$(-1, 1)$	$(-\infty, \infty)$
$\coth^{-1} x$	$(-\infty, -1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$\text{sech}^{-1} x$	$(0, 1]$	$[0, \infty)$
$\text{csch}^{-1} x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$

Table 6.3 Domains and Ranges of the Inverse Hyperbolic Functions

The graphs of the inverse hyperbolic functions are shown in the following figure.

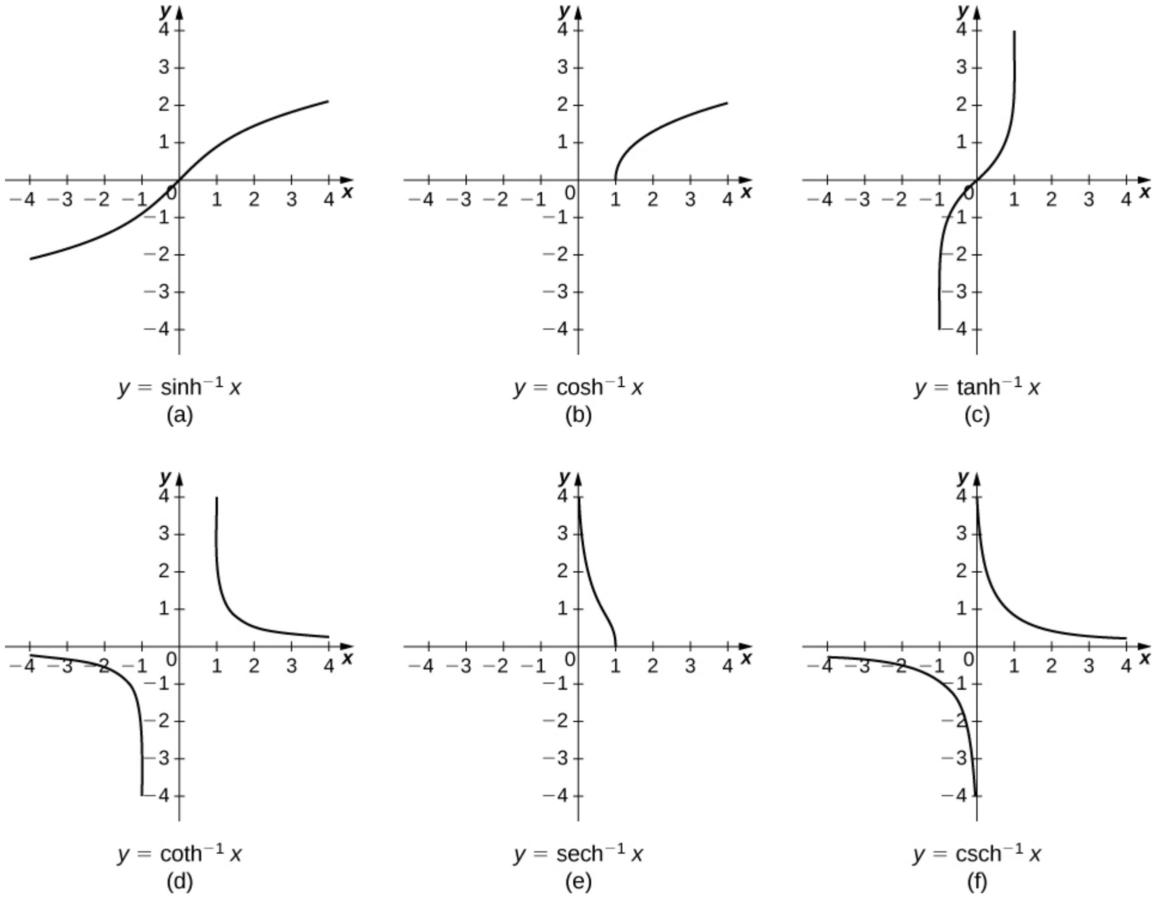


Figure 6.82 Graphs of the inverse hyperbolic functions.

To find the derivatives of the inverse functions, we use implicit differentiation. We have

$$\begin{aligned} y &= \sinh^{-1} x \\ \sinh y &= x \\ \frac{d}{dx}\sinh y &= \frac{d}{dx}x \\ \cosh y\frac{dy}{dx} &= 1. \end{aligned}$$

Recall that $\cosh^2 y - \sinh^2 y = 1$, so $\cosh y = \sqrt{1+\sinh^2 y}$. Then,

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}.$$

We can derive differentiation formulas for the other inverse hyperbolic functions in a similar fashion. These differentiation formulas are summarized in the following table.

$f(x)$	$\frac{d}{dx}f(x)$
$\sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$
$\coth^{-1} x$	$\frac{1}{1-x^2}$
$\operatorname{sech}^{-1} x$	$\frac{-1}{x\sqrt{1-x^2}}$
$\operatorname{csch}^{-1} x$	$\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$

Table 6.4 Derivatives of the Inverse Hyperbolic Functions

Note that the derivatives of $\tanh^{-1} x$ and $\coth^{-1} x$ are the same. Thus, when we integrate $1/(1-x^2)$, we need to select the proper antiderivative based on the domain of the functions and the values of x. Integration formulas involving the inverse hyperbolic functions are summarized as follows.

$$\int \frac{1}{\sqrt{1+u^2}}du = \sinh^{-1} u + C \qquad \int \frac{1}{u\sqrt{1-u^2}}du = -\operatorname{sech}^{-1} |u| + C$$

$$\int \frac{1}{\sqrt{u^2-1}}du = \cosh^{-1} u + C \qquad \int \frac{1}{u\sqrt{1+u^2}}du = -\operatorname{csch}^{-1} |u| + C$$

$$\int \frac{1}{1-u^2}du = \begin{cases} \tanh^{-1} u + C \text{ if } |u| < 1 \\ \coth^{-1} u + C \text{ if } |u| > 1 \end{cases}$$

Example 6.49

Differentiating Inverse Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{3}\right)\right)$

b. $\frac{d}{dx}\left(\tanh^{-1} x\right)^2$

Solution

Using the formulas in **Table 6.4** and the chain rule, we obtain the following results:

a. $\frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{3}\right)\right) = \frac{1}{3\sqrt{1+\frac{x^2}{9}}} = \frac{1}{\sqrt{9+x^2}}$

b. $\frac{d}{dx}\left(\tanh^{-1}x\right)^2 = \frac{2\left(\tanh^{-1}x\right)}{1-x^2}$

6.49 Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\cosh^{-1}(3x)\right)$

b. $\frac{d}{dx}\left(\coth^{-1}x\right)^3$

Example 6.50

Integrals Involving Inverse Hyperbolic Functions

Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{4x^2-1}}dx$

b. $\int \frac{1}{2x\sqrt{1-9x^2}}dx$

Solution

We can use u-substitution in both cases.

a. Let $u = 2x$. Then, $du = 2dx$ and we have

$$\int \frac{1}{\sqrt{4x^2-1}}dx = \int \frac{1}{2\sqrt{u^2-1}}du = \frac{1}{2}\cosh^{-1}u + C = \frac{1}{2}\cosh^{-1}(2x) + C.$$

b. Let $u = 3x$. Then, $du = 3dx$ and we obtain

$$\int \frac{1}{2x\sqrt{1-9x^2}}dx = \frac{1}{2}\int \frac{1}{u\sqrt{1-u^2}}du = -\frac{1}{2}\operatorname{sech}^{-1}|u| + C = -\frac{1}{2}\operatorname{sech}^{-1}|3x| + C.$$

6.50 Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{x^2-4}}dx, \quad x > 2$

b. $\int \frac{1}{\sqrt{1-e^{2x}}}dx$

Applications

One physical application of hyperbolic functions involves hanging cables. If a cable of uniform density is suspended between two supports without any load other than its own weight, the cable forms a curve called a **catenary**. High-voltage power lines, chains hanging between two posts, and strands of a spider's web all form catenaries. The following figure shows chains hanging from a row of posts.

Figure 6.83 Chains between these posts take the shape of a catenary. (credit: modification of work by OKFoundryCompany, Flickr)

Hyperbolic functions can be used to model catenaries. Specifically, functions of the form $y = a\cosh(x/a)$ are catenaries. **Figure 6.84** shows the graph of $y = 2\cosh(x/2)$.

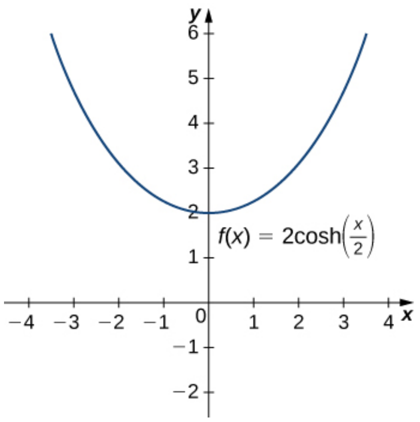


Figure 6.84 A hyperbolic cosine function forms the shape of a catenary.

Example 6.51

Using a Catenary to Find the Length of a Cable

Assume a hanging cable has the shape $10\cosh(x/10)$ for $-15 \le x \le 15,$ where x is measured in feet. Determine the length of the cable (in feet).

Solution

Recall from Section 2.4 that the formula for arc length is

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

We have $f(x) = 10\cosh(x/10),$ so $f'(x) = \sinh(x/10).$ Then

$$\begin{aligned}\text{Arc Length} &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \\ &= \int_{-15}^{15} \sqrt{1 + \sinh^2\left(\frac{x}{10}\right)}\, dx.\end{aligned}$$

Now recall that $1 + \sinh^2 x = \cosh^2 x,$ so we have

$$\begin{aligned}\text{Arc Length} &= \int_{-15}^{15} \sqrt{1 + \sinh^2\left(\frac{x}{10}\right)}\, dx \\ &= \int_{-15}^{15} \cosh\left(\frac{x}{10}\right) dx \\ &= 10\sinh\left(\frac{x}{10}\right)\Big|_{-15}^{15} = 10\left[\sinh\left(\frac{3}{2}\right) - \sinh\left(-\frac{3}{2}\right)\right] = 20\sinh\left(\frac{3}{2}\right) \\ &\approx 42.586\text{ ft.}\end{aligned}$$

6.51 Assume a hanging cable has the shape $15\cosh(x/15)$ for $-20 \le x \le 20.$ Determine the length of the cable (in feet).

6.9 EXERCISES

377. **[T]** Find expressions for $\cosh x + \sinh x$ and $\cosh x - \sinh x$. Use a calculator to graph these functions and ensure your expression is correct.

378. From the definitions of $\cosh(x)$ and $\sinh(x)$, find their antiderivatives.

379. Show that $\cosh(x)$ and $\sinh(x)$ satisfy $y'' = y$.

380. Use the quotient rule to verify that $\tanh(x)' = \text{sech}^2(x)$.

381. Derive $\cosh^2(x) + \sinh^2(x) = \cosh(2x)$ from the definition.

382. Take the derivative of the previous expression to find an expression for $\sinh(2x)$.

383. Prove $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$ by changing the expression to exponentials.

384. Take the derivative of the previous expression to find an expression for $\cosh(x + y)$.

For the following exercises, find the derivatives of the given functions and graph along with the function to ensure your answer is correct.

385. **[T]** $\cosh(3x + 1)$

386. **[T]** $\sinh\left(x^2\right)$

387. **[T]** $\dfrac{1}{\cosh(x)}$

388. **[T]** $\sinh(\ln(x))$

389. **[T]** $\cosh^2(x) + \sinh^2(x)$

390. **[T]** $\cosh^2(x) - \sinh^2(x)$

391. **[T]** $\tanh\left(\sqrt{x^2 + 1}\right)$

392. **[T]** $\dfrac{1 + \tanh(x)}{1 - \tanh(x)}$

393. **[T]** $\sinh^6(x)$

394. **[T]** $\ln(\text{sech}(x) + \tanh(x))$

For the following exercises, find the antiderivatives for the given functions.

395. $\cosh(2x + 1)$

396. $\tanh(3x + 2)$

397. $x\cosh\left(x^2\right)$

398. $3x^3\tanh\left(x^4\right)$

399. $\cosh^2(x)\sinh(x)$

400. $\tanh^2(x)\text{sech}^2(x)$

401. $\dfrac{\sinh(x)}{1 + \cosh(x)}$

402. $\coth(x)$

403. $\cosh(x) + \sinh(x)$

404. $(\cosh(x) + \sinh(x))^n$

For the following exercises, find the derivatives for the functions.

405. $\tanh^{-1}(4x)$

406. $\sinh^{-1}\left(x^2\right)$

407. $\sinh^{-1}(\cosh(x))$

408. $\cosh^{-1}\left(x^3\right)$

409. $\tanh^{-1}(\cos(x))$

410. $e^{\sinh^{-1}(x)}$

411. $\ln\left(\tanh^{-1}(x)\right)$

For the following exercises, find the antiderivatives for the functions.

412. $\displaystyle\int \frac{dx}{4 - x^2}$

413. $\displaystyle\int \frac{dx}{a^2 - x^2}$

414. $\int \frac{dx}{\sqrt{x^2+1}}$

415. $\int \frac{x\,dx}{\sqrt{x^2+1}}$

416. $\int -\frac{dx}{x\sqrt{1-x^2}}$

417. $\int \frac{e^x}{\sqrt{e^{2x}-1}}$

418. $\int -\frac{2x}{x^4-1}$

For the following exercises, use the fact that a falling body with friction equal to velocity squared obeys the equation $dv/dt = g - v^2$.

419. Show that $v(t) = \sqrt{g}\tanh((\sqrt{g})t)$ satisfies this equation.

420. Derive the previous expression for $v(t)$ by integrating $\frac{dv}{g - v^2} = dt$.

421. **[T]** Estimate how far a body has fallen in 12 seconds by finding the area underneath the curve of $v(t)$.

For the following exercises, use this scenario: A cable hanging under its own weight has a slope $S = dy/dx$ that satisfies $dS/dx = c\sqrt{1+S^2}$. The constant c is the ratio of cable density to tension.

422. Show that $S = \sinh(cx)$ satisfies this equation.

423. Integrate $dy/dx = \sinh(cx)$ to find the cable height $y(x)$ if $y(0) = 1/c$.

424. Sketch the cable and determine how far down it sags at $x = 0$.

For the following exercises, solve each problem.

425. **[T]** A chain hangs from two posts 2 m apart to form a catenary described by the equation $y = 2\cosh(x/2) - 1$. Find the slope of the catenary at the left fence post.

426. **[T]** A chain hangs from two posts four meters apart to form a catenary described by the equation $y = 4\cosh(x/4) - 3$. Find the total length of the catenary (arc length).

427. **[T]** A high-voltage power line is a catenary described by $y = 10\cosh(x/10)$. Find the ratio of the area under the catenary to its arc length. What do you notice?

428. A telephone line is a catenary described by $y = a\cosh(x/a)$. Find the ratio of the area under the catenary to its arc length. Does this confirm your answer for the previous question?

429. Prove the formula for the derivative of $y = \sinh^{-1}(x)$ by differentiating $x = \sinh(y)$. (*Hint:* Use hyperbolic trigonometric identities.)

430. Prove the formula for the derivative of $y = \cosh^{-1}(x)$ by differentiating $x = \cosh(y)$. (*Hint:* Use hyperbolic trigonometric identities.)

431. Prove the formula for the derivative of $y = \operatorname{sech}^{-1}(x)$ by differentiating $x = \operatorname{sech}(y)$. (*Hint:* Use hyperbolic trigonometric identities.)

432. Prove that $(\cosh(x) + \sinh(x))^n = \cosh(nx) + \sinh(nx)$.

433. Prove the expression for $\sinh^{-1}(x)$. Multiply $x = \sinh(y) = (1/2)\left(e^y - e^{-y}\right)$ by $2e^y$ and solve for y. Does your expression match the textbook?

434. Prove the expression for $\cosh^{-1}(x)$. Multiply $x = \cosh(y) = (1/2)\left(e^y - e^{-y}\right)$ by $2e^y$ and solve for y. Does your expression match the textbook?

CHAPTER 6 REVIEW

KEY TERMS

arc length the arc length of a curve can be thought of as the distance a person would travel along the path of the curve

catenary a curve in the shape of the function $y = a\cosh(x/a)$ is a catenary; a cable of uniform density suspended between two supports assumes the shape of a catenary

center of mass the point at which the total mass of the system could be concentrated without changing the moment

centroid the centroid of a region is the geometric center of the region; laminas are often represented by regions in the plane; if the lamina has a constant density, the center of mass of the lamina depends only on the shape of the corresponding planar region; in this case, the center of mass of the lamina corresponds to the centroid of the representative region

cross-section the intersection of a plane and a solid object

density function a density function describes how mass is distributed throughout an object; it can be a linear density, expressed in terms of mass per unit length; an area density, expressed in terms of mass per unit area; or a volume density, expressed in terms of mass per unit volume; weight-density is also used to describe weight (rather than mass) per unit volume

disk method a special case of the slicing method used with solids of revolution when the slices are disks

doubling time if a quantity grows exponentially, the doubling time is the amount of time it takes the quantity to double, and is given by $(\ln 2)/k$

exponential decay systems that exhibit exponential decay follow a model of the form $y = y_0 e^{-kt}$

exponential growth systems that exhibit exponential growth follow a model of the form $y = y_0 e^{kt}$

frustum a portion of a cone; a frustum is constructed by cutting the cone with a plane parallel to the base

half-life if a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by $(\ln 2)/k$

Hooke's law this law states that the force required to compress (or elongate) a spring is proportional to the distance the spring has been compressed (or stretched) from equilibrium; in other words, $F = kx$, where k is a constant

hydrostatic pressure the pressure exerted by water on a submerged object

lamina a thin sheet of material; laminas are thin enough that, for mathematical purposes, they can be treated as if they are two-dimensional

method of cylindrical shells a method of calculating the volume of a solid of revolution by dividing the solid into nested cylindrical shells; this method is different from the methods of disks or washers in that we integrate with respect to the opposite variable

moment if n masses are arranged on a number line, the moment of the system with respect to the origin is given by $M = \sum_{i=1}^{n} m_i x_i$; if, instead, we consider a region in the plane, bounded above by a function $f(x)$ over an interval $[a, b]$, then the moments of the region with respect to the x- and y-axes are given by $M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx$ and $M_y = \rho \int_a^b x f(x) dx$, respectively

slicing method a method of calculating the volume of a solid that involves cutting the solid into pieces, estimating the volume of each piece, then adding these estimates to arrive at an estimate of the total volume; as the number of slices goes to infinity, this estimate becomes an integral that gives the exact value of the volume

solid of revolution a solid generated by revolving a region in a plane around a line in that plane

surface area the surface area of a solid is the total area of the outer layer of the object; for objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces

symmetry principle the symmetry principle states that if a region R is symmetric about a line l, then the centroid of R lies on l

theorem of Pappus for volume this theorem states that the volume of a solid of revolution formed by revolving a region around an external axis is equal to the area of the region multiplied by the distance traveled by the centroid of the region

washer method a special case of the slicing method used with solids of revolution when the slices are washers

work the amount of energy it takes to move an object; in physics, when a force is constant, work is expressed as the product of force and distance

KEY EQUATIONS

- **Area between two curves, integrating on the *x*-axis**

  $$A = \int_a^b [f(x) - g(x)]dx$$

- **Area between two curves, integrating on the *y*-axis**

  $$A = \int_c^d [u(y) - v(y)]dy$$

- **Disk Method along the *x*-axis**

  $$V = \int_a^b \pi[f(x)]^2\, dx$$

- **Disk Method along the *y*-axis**

  $$V = \int_c^d \pi[g(y)]^2\, dy$$

- **Washer Method**

  $$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx$$

- **Method of Cylindrical Shells**

  $$V = \int_a^b (2\pi x f(x))dx$$

- **Arc Length of a Function of *x***

  $$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

- **Arc Length of a Function of *y***

  $$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

- **Surface Area of a Function of *x***

  $$\text{Surface Area} = \int_a^b \left(2\pi f(x)\sqrt{1 + (f'(x))^2}\right)dx$$

- **Mass of a one-dimensional object**

  $$m = \int_a^b \rho(x)dx$$

- **Mass of a circular object**

  $$m = \int_0^r 2\pi x\rho(x)dx$$

- **Work done on an object**
  $W = \int_a^b F(x)dx$
- **Hydrostatic force on a plate**
  $F = \int_a^b \rho w(x)s(x)dx$
- **Mass of a lamina**
  $m = \rho \int_a^b f(x)dx$
- **Moments of a lamina**
  $M_x = \rho \int_a^b \frac{[f(x)]^2}{2}dx$ and $M_y = \rho \int_a^b xf(x)dx$
- **Center of mass of a lamina**
  $\bar{x} = \frac{M_y}{m}$ and $\bar{y} = \frac{M_x}{m}$
- **Natural logarithm function**
- $\ln x = \int_1^x \frac{1}{t}dt$ Z
- **Exponential function** $y = e^x$
- $\ln y = \ln(e^x) = x$ Z

KEY CONCEPTS

6.1 Areas between Curves

- Just as definite integrals can be used to find the area under a curve, they can also be used to find the area between two curves.
- To find the area between two curves defined by functions, integrate the difference of the functions.
- If the graphs of the functions cross, or if the region is complex, use the absolute value of the difference of the functions. In this case, it may be necessary to evaluate two or more integrals and add the results to find the area of the region.
- Sometimes it can be easier to integrate with respect to y to find the area. The principles are the same regardless of which variable is used as the variable of integration.

6.2 Determining Volumes by Slicing

- Definite integrals can be used to find the volumes of solids. Using the slicing method, we can find a volume by integrating the cross-sectional area.
- For solids of revolution, the volume slices are often disks and the cross-sections are circles. The method of disks involves applying the method of slicing in the particular case in which the cross-sections are circles, and using the formula for the area of a circle.
- If a solid of revolution has a cavity in the center, the volume slices are washers. With the method of washers, the area of the inner circle is subtracted from the area of the outer circle before integrating.

6.3 Volumes of Revolution: Cylindrical Shells

- The method of cylindrical shells is another method for using a definite integral to calculate the volume of a solid of revolution. This method is sometimes preferable to either the method of disks or the method of washers because we integrate with respect to the other variable. In some cases, one integral is substantially more complicated than the other.

- The geometry of the functions and the difficulty of the integration are the main factors in deciding which integration method to use.

6.4 Arc Length of a Curve and Surface Area

- The arc length of a curve can be calculated using a definite integral.
- The arc length is first approximated using line segments, which generates a Riemann sum. Taking a limit then gives us the definite integral formula. The same process can be applied to functions of y.
- The concepts used to calculate the arc length can be generalized to find the surface area of a surface of revolution.
- The integrals generated by both the arc length and surface area formulas are often difficult to evaluate. It may be necessary to use a computer or calculator to approximate the values of the integrals.

6.5 Physical Applications

- Several physical applications of the definite integral are common in engineering and physics.
- Definite integrals can be used to determine the mass of an object if its density function is known.
- Work can also be calculated from integrating a force function, or when counteracting the force of gravity, as in a pumping problem.
- Definite integrals can also be used to calculate the force exerted on an object submerged in a liquid.

6.6 Moments and Centers of Mass

- Mathematically, the center of mass of a system is the point at which the total mass of the system could be concentrated without changing the moment. Loosely speaking, the center of mass can be thought of as the balancing point of the system.
- For point masses distributed along a number line, the moment of the system with respect to the origin is $M = \sum_{i=1}^{n} m_i x_i$. For point masses distributed in a plane, the moments of the system with respect to the x- and y-axes, respectively, are $M_x = \sum_{i=1}^{n} m_i y_i$ and $M_y = \sum_{i=1}^{n} m_i x_i$, respectively.
- For a lamina bounded above by a function $f(x)$, the moments of the system with respect to the x- and y-axes, respectively, are $M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx$ and $M_y = \rho \int_a^b x f(x) dx$.
- The x- and y-coordinates of the center of mass can be found by dividing the moments around the y-axis and around the x-axis, respectively, by the total mass. The symmetry principle says that if a region is symmetric with respect to a line, then the centroid of the region lies on the line.
- The theorem of Pappus for volume says that if a region is revolved around an external axis, the volume of the resulting solid is equal to the area of the region multiplied by the distance traveled by the centroid of the region.

6.7 Integrals, Exponential Functions, and Logarithms

- The earlier treatment of logarithms and exponential functions did not define the functions precisely and formally. This section develops the concepts in a mathematically rigorous way.
- The cornerstone of the development is the definition of the natural logarithm in terms of an integral.
- The function e^x is then defined as the inverse of the natural logarithm.
- General exponential functions are defined in terms of e^x, and the corresponding inverse functions are general logarithms.

- Familiar properties of logarithms and exponents still hold in this more rigorous context.

6.8 Exponential Growth and Decay

- Exponential growth and exponential decay are two of the most common applications of exponential functions.
- Systems that exhibit exponential growth follow a model of the form $y = y_0 e^{kt}$.
- In exponential growth, the rate of growth is proportional to the quantity present. In other words, $y' = ky$.
- Systems that exhibit exponential growth have a constant doubling time, which is given by $(\ln 2)/k$.
- Systems that exhibit exponential decay follow a model of the form $y = y_0 e^{-kt}$.
- Systems that exhibit exponential decay have a constant half-life, which is given by $(\ln 2)/k$.

6.9 Calculus of the Hyperbolic Functions

- Hyperbolic functions are defined in terms of exponential functions.
- Term-by-term differentiation yields differentiation formulas for the hyperbolic functions. These differentiation formulas give rise, in turn, to integration formulas.
- With appropriate range restrictions, the hyperbolic functions all have inverses.
- Implicit differentiation yields differentiation formulas for the inverse hyperbolic functions, which in turn give rise to integration formulas.
- The most common physical applications of hyperbolic functions are calculations involving catenaries.

CHAPTER 6 REVIEW EXERCISES

True or False? Justify your answer with a proof or a counterexample.

435. The amount of work to pump the water out of a half-full cylinder is half the amount of work to pump the water out of the full cylinder.

436. If the force is constant, the amount of work to move an object from $x = a$ to $x = b$ is $F(b - a)$.

437. The disk method can be used in any situation in which the washer method is successful at finding the volume of a solid of revolution.

438. If the half-life of seaborgium-266 is 360 ms, then $k = (\ln(2))/360$.

For the following exercises, use the requested method to determine the volume of the solid.

439. The volume that has a base of the ellipse $x^2/4 + y^2/9 = 1$ and cross-sections of an equilateral triangle perpendicular to the y-axis. Use the method of slicing.

440. $y = x^2 - x$, from $x = 1$ to $x = 4$, rotated around they-axis using the washer method

441. $x = y^2$ and $x = 3y$ rotated around the y-axis using the washer method

442. $x = 2y^2 - y^3$, $x = 0$, and $y = 0$ rotated around the x-axis using cylindrical shells

For the following exercises, find

a. the area of the region,
b. the volume of the solid when rotated around the x-axis, and
c. the volume of the solid when rotated around the y-axis. Use whichever method seems most appropriate to you.

443. $y = x^3$, $x = 0$, $y = 0$, and $x = 2$

444. $y = x^2 - x$ and $x = 0$

445. **[T]** $y = \ln(x) + 2$ and $y = x$

446. $y = x^2$ and $y = \sqrt{x}$

447. $y = 5 + x$, $y = x^2$, $x = 0$, and $x = 1$

448. Below $x^2 + y^2 = 1$ and above $y = 1 - x$

449. Find the mass of $\rho = e^{-x}$ on a disk centered at the origin with radius 4.

450. Find the center of mass for $\rho = \tan^2 x$ on $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.

451. Find the mass and the center of mass of $\rho = 1$ on the region bounded by $y = x^5$ and $y = \sqrt{x}$.

For the following exercises, find the requested arc lengths.

452. The length of x for $y = \cosh(x)$ from $x = 0$ to $x = 2$.

453. The length of y for $x = 3 - \sqrt{y}$ from $y = 0$ to $y = 4$

For the following exercises, find the surface area and volume when the given curves are revolved around the specified axis.

454. The shape created by revolving the region between $y = 4 + x$, $y = 3 - x$, $x = 0$, and $x = 2$ rotated around the y-axis.

455. The loudspeaker created by revolving $y = 1/x$ from $x = 1$ to $x = 4$ around the x-axis.

For the following exercises, consider the Karun-3 dam in Iran. Its shape can be approximated as an isosceles triangle with height 205 m and width 388 m. Assume the current depth of the water is 180 m. The density of water is 1000 kg/m ^{3}.

456. Find the total force on the wall of the dam.

457. You are a crime scene investigator attempting to determine the time of death of a victim. It is noon and $45°\text{F}$ outside and the temperature of the body is $78°\text{F}$. You know the cooling constant is $k = 0.00824°\text{F/min}$. When did the victim die, assuming that a human's temperature is $98°\text{F}$?

For the following exercise, consider the stock market crash in 1929 in the United States. The table lists the Dow Jones industrial average per year leading up to the crash.

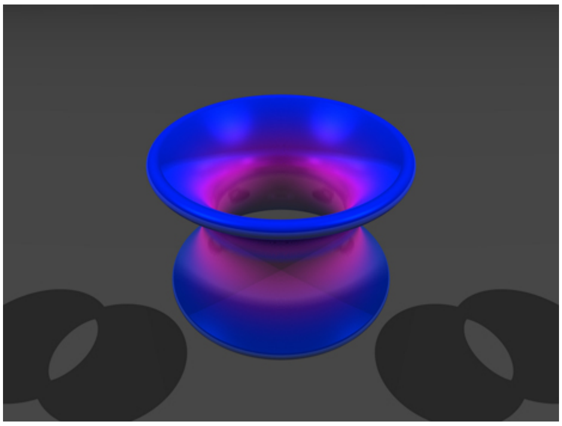
Years after 1920	Value ($)
1	63.90
3	100
5	110
7	160
9	381.17

Source: http://stockcharts.com/freecharts/historical/djia19201940.html

458. **[T]** The best-fit exponential curve to these data is given by $y = 40.71 + 1.224^x$. Why do you think the gains of the market were unsustainable? Use first and second derivatives to help justify your answer. What would this model predict the Dow Jones industrial average to be in 2014 ?

For the following exercises, consider the catenoid, the only solid of revolution that has a minimal surface, or zero mean curvature. A catenoid in nature can be found when stretching soap between two rings.

459. Find the volume of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the x-axis, as shown here.



460. Find surface area of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the x -axis.